# *Imperfections in Crystalline Solids 1st edition Cai Solutions Manual*

SKU: 9781107123137-SOLUTIONS

# Chapter 2

# Stress, Strain and Isotropic Elasticity

## 2.7 Exercise Problems

**2.1** A strip of metal is subjected to a uniform tension stress $\sigma_{yy} = \sigma$, as shown in Fig. 2.16.

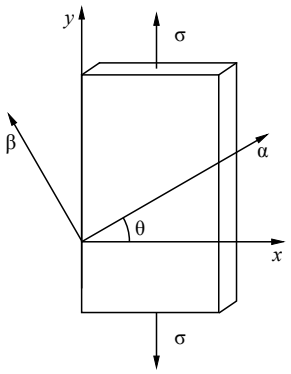



Figure 2.16: A strip of metal subjected to tensile stress.

(a) Calculate the shear stress $\sigma_{\alpha\beta}$ acting on a plane that is 45° to the horizontal axis.

(b) Write a general expression for the shear stress $\sigma_{\alpha\beta}$ on a plane making an arbitrary angle $\theta$ with the horizontal axis.

(c) Show that $\sigma_{\alpha\beta}$ is maximum at $\theta = 45°$.

(d) Calculate the hydrostatic pressure $p$ in the strip.

**Solution**

(a) $\sigma_{\alpha\beta} = (\cos 45°)(\sin 45°)\,\sigma_{yy} = \frac{1}{2}\,\sigma$.

(b) $\sigma_{\alpha\beta} = (\cos\theta)(\sin\theta)\,\sigma_{yy} = \frac{1}{2}\sin(2\theta)\,\sigma$.

(c) Maximum of $\sin(2\theta)$ occurs at $\theta = 45°$ where $\sin(2\theta) = \sin 90° = 1$. Therefore, maximum of $\sigma_{\alpha\beta}$ occurs at $\theta = 45°$ where $\sigma_{\alpha\beta} = \frac{1}{2}\,\sigma$.



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(d) $p = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = -\frac{1}{3}\,\sigma$.

**2.2** An FCC single crystal thin film with the orientation shown in Fig. 2.17 is subjected to an equal-biaxial tensile stress $\sigma$. Plastic deformation of this crystal will occur by slip on the $(1\,\bar{1}\,1)$ plane (shaded) and along the $[0\,1\,1]$ direction as shown. Write an expression for the shear stress acting on the slip plane in the slip direction.

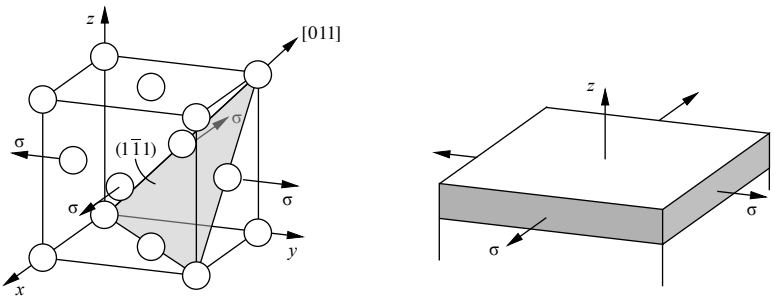



Figure 2.17: An FCC single crystal thin film under biaxial tension.

**Solution**
In $(x\,y\,z)$ coordinate system,

$$\sigma_{xx} = \sigma_{yy} = \sigma\,, \quad \sigma_{zz} = 0$$

Choose a new coordinate system $(x'\,y'\,z')$ in which the $x'$-, $y'$-, and $z'$-axes are along the $[0\,1\,1]$, $[1\,\bar{1}\,1]$, and $[2\,1\,\bar{1}]$ directions respectively. The stress component in question is $\sigma_{x'y'}$.

$$\begin{aligned}\sigma_{x'y'} &= Q_{1k}\,Q_{2l}\,\sigma_{kl} \\ &= Q_{11}\,Q_{21}\,\sigma_{xx} + Q_{12}\,Q_{22}\,\sigma_{yy}\end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$\begin{aligned}Q_{11} &= \frac{1}{\sqrt{2}}[0\,1\,1]\cdot[1\,0\,0] = 0 \\ Q_{21} &= \frac{1}{\sqrt{3}}[1\,\bar{1}\,1]\cdot[1\,0\,0] = \frac{1}{\sqrt{3}} \\ Q_{12} &= \frac{1}{\sqrt{2}}[0\,1\,1]\cdot[0\,1\,0] = \frac{1}{\sqrt{2}} \\ Q_{22} &= \frac{1}{\sqrt{3}}[1\,\bar{1}\,1]\cdot[0\,1\,0] = -\frac{1}{\sqrt{3}}\end{aligned}$$

Therefore,

$$\sigma_{x'y'} = \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{3}}\right)\sigma = -\frac{1}{\sqrt{6}}\,\sigma$$

**2.3** A single crystal of BCC iron is pulled in uniaxial tension along a cube direction, $[1\,0\,0]$. We assume slip occurs on the $(\bar{2}\,1\,1)$ plane and in the $[1\,1\,1]$ direction as shown in Fig. 2.18(a). The

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critical resolved shear stress (CRSS) for slip to occur is 10 MPa. Calculate the tensile stress at which plastic yielding begins.

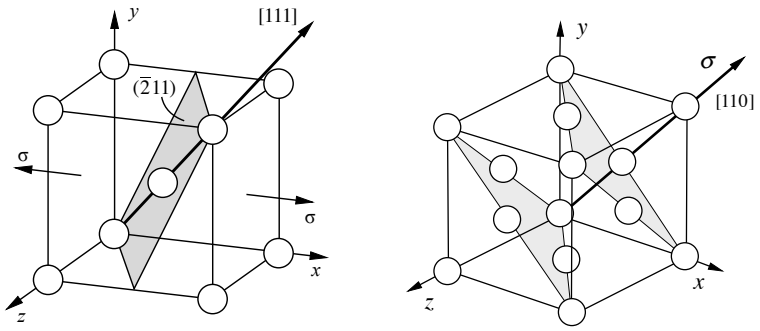



Figure 2.18: (a) A single crystal of BCC iron subjected to tension along the [1 0 0] direction. (b) An FCC single crystal subjected to tension along the [1 1 0] direction. Two $(1\,1\,\bar{1})$ planes are shaded to help show the atomic positions.

**Solution**

In $(x\,y\,z)$ coordinate system,

$$\sigma_{xx} = \sigma\,, \quad \sigma_{zz} = \sigma_{yy} = 0$$

Choose a new coordinate system $(x'\,y'\,z')$ in which the $x'$-, $y'$-, and $z'$-axes are along the $[1\,1\,1]$, $[\bar{2}\,1\,1]$, and $[0\,\bar{1}\,1]$ directions respectively. The stress component for the CRSS is $\sigma_{x'y'}$.

$$\begin{aligned}\sigma_{x'y'} &= Q_{1k}\,Q_{2l}\,\sigma_{kl} \\ &= Q_{11}\,Q_{21}\,\sigma_{xx}\end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$Q_{11} = \frac{1}{\sqrt{3}}[1\,1\,1]\cdot[1\,0\,0] = \frac{1}{\sqrt{3}}$$
$$Q_{21} = \frac{1}{\sqrt{6}}[\bar{2}\,1\,1]\cdot[1\,0\,0] = -\frac{2}{\sqrt{6}}$$

Therefore,

$$\sigma_{x'y'} = \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{2}{\sqrt{6}}\right)\sigma = -\frac{\sqrt{2}}{3}\,\sigma$$

$$|\sigma_{x'y'}| = 10\,\text{MPa}\sigma \quad = \frac{3}{\sqrt{2}} \times 10\,\text{MPa} = 21.2\,\text{MPa}$$

**2.4** An FCC single crystal is subjected to a uniform tensile stress of magnitude $\sigma$ in the $[1\,1\,0]$ direction, as shown in Fig. 2.18(b). Calculate all the non-zero stress components in the cubic $(xyz)$ axes.

**Solution**

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Let the $x$, $y$, $z$ axes in the $(x\,y\,z)$ coordinate system be aligned along the $[1\,0\,0]$, $[0\,1\,0]$, and $[0\,0\,1]$ directions, respectively. Choose a new coordinate system $(x'\,y'\,z')$ in which the $x'$-, $y'$-, and $z'$-axes are along the $[1\,1\,0]$, $[\bar{1}\,1\,1]$, and $[1\,\bar{1}\,2]$ directions respectively. Here we know that

$$\sigma_{x'x'} = \sigma\,, \quad \sigma_{y'y'} = \sigma_{z'z'} = 0$$

The stress components in the $(x\,y\,z)$ coordinate system are

$$\begin{aligned}\sigma_{ij} &= Q^{\mathrm{T}}_{ik}\, Q^{\mathrm{T}}_{jl}\, \sigma'_{kl} \\ &= Q^{\mathrm{T}}_{i1}\, Q^{\mathrm{T}}_{j1}\, \sigma_{x'x'}\end{aligned}$$

where $Q^{\mathrm{T}}_{ik} = (\hat{e}'_k \cdot \hat{e}_i)$.

$$\begin{aligned}Q^{\mathrm{T}}_{11} &= \frac{1}{\sqrt{2}}[1\,1\,0]\cdot[1\,0\,0] = \frac{1}{\sqrt{2}} \\ Q^{\mathrm{T}}_{21} &= \frac{1}{\sqrt{2}}[1\,1\,0]\cdot[0\,1\,0] = \frac{1}{\sqrt{2}} \\ Q^{\mathrm{T}}_{31} &= \frac{1}{\sqrt{2}}[1\,1\,0]\cdot[0\,0\,1] = 0\end{aligned}$$

Therefore,

$$\begin{aligned}\sigma_{xx} &= Q^{\mathrm{T}}_{11}\, Q^{\mathrm{T}}_{11}\, \sigma_{x'x'} = \frac{1}{2}\,\sigma \\ \sigma_{yy} &= Q^{\mathrm{T}}_{21}\, Q^{\mathrm{T}}_{21}\, \sigma_{x'x'} = \frac{1}{2}\,\sigma \\ \sigma_{xy} &= Q^{\mathrm{T}}_{11}\, Q^{\mathrm{T}}_{21}\, \sigma_{x'x'} = \frac{1}{2}\,\sigma \\ \sigma_{xz} &= \sigma_{yz} = \sigma_{zz} = 0\end{aligned}$$

**2.5** Consider a single crystal rod subjected to a uniaxial tensile stress $\sigma$, as shown in Fig. 2.19. Plastic deformation would occur if the resolved shear stress $\tau$ on a certain crystallographic plane (with normal vector $\hat{n}$) and along a certain crystallographic direction $\hat{m}$ exceeds a threshold value. Express the resolved shear stress $\tau$ in terms of the tensile stress $\sigma$ and the angles $\theta$ and $\phi$ that vectors $\hat{n}$ and $\hat{m}$ make with the tensile axis. The ratio $S = \tau/\sigma$ is called the *Schmid factor*.

**Solution**

$$\begin{aligned}\tau &= \cos\theta\,\cos\phi\,\sigma \\ S &= \cos\theta\,\cos\phi\end{aligned}$$

**2.6** The stress field of an infinite isotropic elastic medium containing a pressurized cylindrical hole (of radius $r_0$) has the following form,

$$\begin{aligned}\sigma_{rr} &= -\frac{A}{r^2} \\ \sigma_{\theta\theta} &= \frac{A}{r^2} \\ \sigma_{r\theta} &= 0\end{aligned} \tag{2.81}$$

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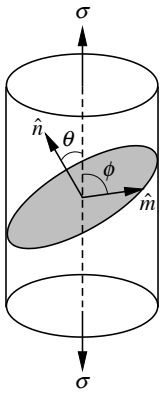



Figure 2.19: Single crystal subjected to uniaxial tension.

for $r \geq r_0$. Transform the stress field into Cartesian components: $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$. Make contour plots of $\sigma_{rr}$, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ and $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$.

**Solution**
Let coordinate system A be the cylindrical ($r$, $\theta$, $z$) coordinate system and coordinate system A' be the Cartesian ($x$, $y$, $z$) coordinate system. Let matrix $Q$ be,

$$Q_{ik} = (\hat{e}_i^{xyz} \cdot \hat{e}_k^{r\theta z}) = \begin{bmatrix} (\hat{e}_x \cdot \hat{e}_r) & (\hat{e}_x \cdot \hat{e}_\theta) & (\hat{e}_x \cdot \hat{e}_z) \\ (\hat{e}_y \cdot \hat{e}_r) & (\hat{e}_y \cdot \hat{e}_\theta) & (\hat{e}_y \cdot \hat{e}_z) \\ (\hat{e}_z \cdot \hat{e}_r) & (\hat{e}_z \cdot \hat{e}_\theta) & (\hat{e}_z \cdot \hat{e}_z) \end{bmatrix}$$

Given that

$$\hat{e}_r = \hat{e}_x \cos\theta + \hat{e}_y \sin\theta$$
$$\hat{e}_\theta = -\hat{e}_x \sin\theta + \hat{e}_y \cos\theta$$

we have,

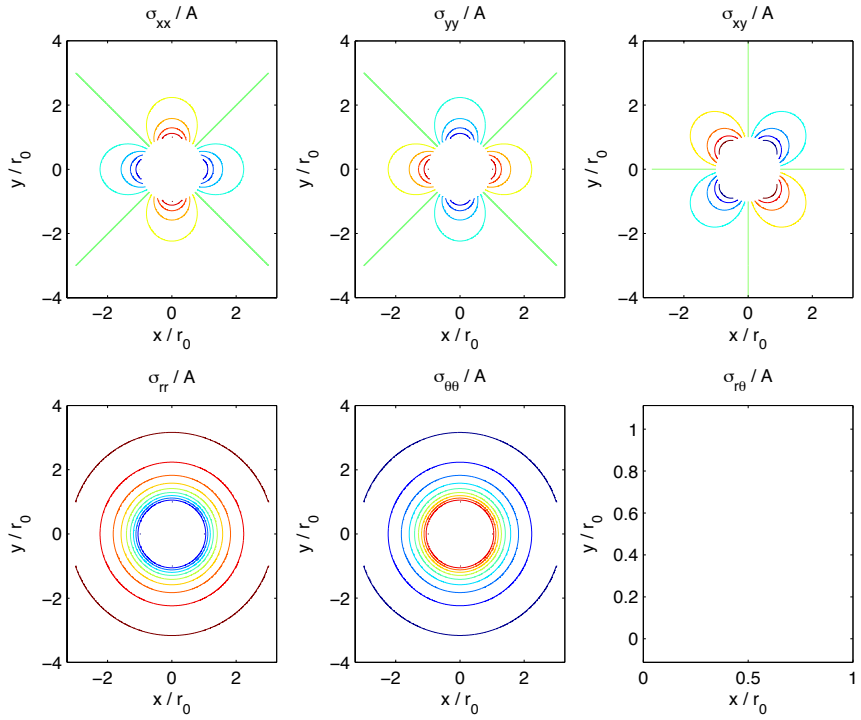
$$Q_{ik} = \begin{bmatrix} Q_{xr} & Q_{x\theta} & Q_{xz} \\ Q_{yr} & Q_{y\theta} & Q_{yz} \\ Q_{zr} & Q_{z\theta} & Q_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
\sigma_{xx} &= Q_{xr}\,Q_{xr}\,\sigma_{rr} + Q_{x\theta}\,Q_{x\theta}\,\sigma_{\theta\theta} = -(\cos^2\theta - \sin^2\theta)\left(\frac{A}{r^2}\right) \\
\sigma_{yy} &= Q_{yr}\,Q_{yr}\,\sigma_{rr} + Q_{y\theta}\,Q_{y\theta}\,\sigma_{\theta\theta} = (\cos^2\theta - \sin^2\theta)\left(\frac{A}{r^2}\right) \\
\sigma_{zz} &= Q_{zr}\,Q_{zr}\,\sigma_{rr} + Q_{z\theta}\,Q_{z\theta}\,\sigma_{\theta\theta} = 0 \\
\sigma_{xy} &= Q_{xr}\,Q_{yr}\,\sigma_{rr} + Q_{x\theta}\,Q_{y\theta}\,\sigma_{\theta\theta} = -2\sin\theta\cos\theta\left(\frac{A}{r^2}\right) \\
\sigma_{yz} &= Q_{yr}\,Q_{zr}\,\sigma_{rr} + Q_{y\theta}\,Q_{z\theta}\,\sigma_{\theta\theta} = 0 \\
\sigma_{xz} &= Q_{xr}\,Q_{zr}\,\sigma_{rr} + Q_{x\theta}\,Q_{z\theta}\,\sigma_{\theta\theta} = 0
\end{aligned}$$

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The results can be simplified as: $\sigma_{xx} = -A\cos(2\theta)/r^2$, $\sigma_{yy} = A\cos(2\theta)/r^2$, $\sigma_{xy} = -A\sin(2\theta)/r^2$. They can be written in terms of $x$, $y$, $z$ as

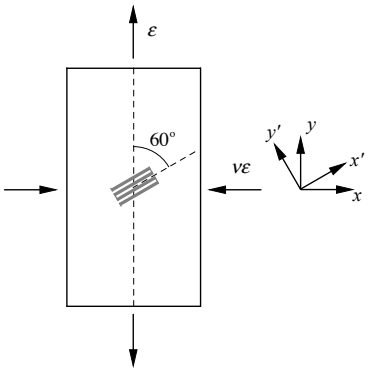
$$\sigma_{xx} = -A\frac{x^2 - y^2}{(x^2 + y^2)^2}$$
$$\sigma_{yy} = \quad A\frac{x^2 - y^2}{(x^2 + y^2)^2}$$
$$\sigma_{xy} = -A\frac{2xy}{(x^2 + y^2)^2}$$




**2.7** A strain gauge measures tensile strain along a given direction through the change of electrical resistance of thin wires aligned in that direction. Consider a solid subjected to uniaxial tensile stress, which induces a normal strain $\varepsilon$ in the tensile direction, and a normal strain $-\nu\varepsilon$ in the perpendicular directions, as shown in Fig. 2.20(a). Express the strain measured by a strain gauge oriented at 60° from the tensile axis.

**Solution**

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For the coordinate system $(x\,y)$ shown,

$$\varepsilon_{xx} = -\nu\,\varepsilon$$
$$\varepsilon_{yy} = \varepsilon$$

The coordinate system $(x'\,y')$ is rotated from the coordinate system $(x\,y)$ by 30°. What the strain gauge measures is component $\varepsilon_{x'x'}$.

$$\begin{aligned}\varepsilon_{x'x'} &= Q_{1k}\,Q_{1l}\,\varepsilon_{kl} \\ &= Q_{11}\,Q_{11}\,\varepsilon_{xx} + Q_{12}\,Q_{12}\,\varepsilon_{yy}\end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

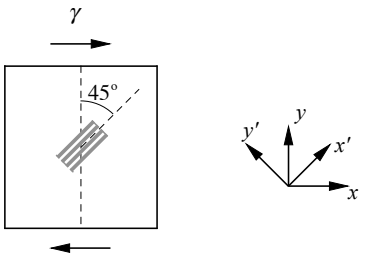
$$Q_{11} = \cos 30^\circ = \frac{\sqrt{3}}{2}$$
$$Q_{12} = \cos 60^\circ = \frac{1}{2}$$

Therefore,

$$\begin{aligned}\varepsilon_{x'x'} &= \frac{\sqrt{3}}{2}\,\frac{\sqrt{3}}{2}\,(-\nu\,\varepsilon) + \frac{1}{2}\,\frac{1}{2}\,\varepsilon \\ &= \frac{1-3\nu}{4}\,\varepsilon\end{aligned}$$

**2.8** Express the strain measured by a strain gauge oriented at 45° from the vertical axis of a solid subjected to an engineering shear strain of $\gamma$, as shown in Fig. 2.20(b).

**Solution**




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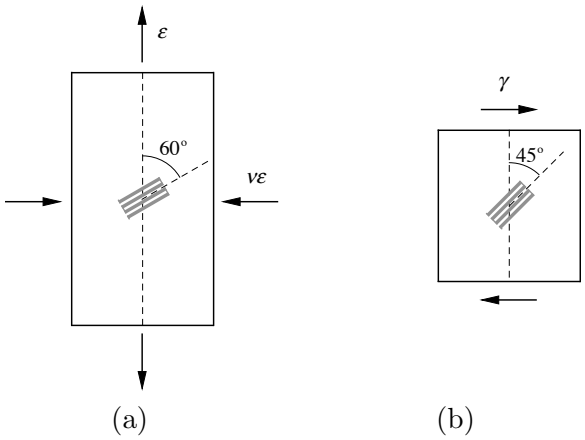



Figure 2.20: Strain gauge.

For the coordinate system $(x\,y)$ shown,

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{\gamma}{2}$$

The coordinate system $(x'\,y')$ is rotated from the coordinate system $(x\,y)$ by 45°. What the strain gauge measures is component $\varepsilon_{x'x'}$.

$$\begin{aligned}\varepsilon_{x'x'} &= Q_{1k}\,Q_{1l}\,\varepsilon_{kl} \\ &= Q_{11}\,Q_{12}\,\varepsilon_{xy} + Q_{12}\,Q_{11}\,\varepsilon_{yx}\end{aligned}$$

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$\begin{aligned}Q_{11} &= \cos 45^\circ = \frac{\sqrt{2}}{2} \\ Q_{12} &= \cos 45^\circ = \frac{\sqrt{2}}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}\varepsilon_{x'x'} &= \frac{\sqrt{2}}{2}\,\frac{\sqrt{2}}{2}\,\frac{\gamma}{2} + \frac{\sqrt{2}}{2}\,\frac{\sqrt{2}}{2}\,\frac{\gamma}{2} \\ &= \frac{\gamma}{2}\end{aligned}$$

**2.9** A thin film is subjected to strain $\varepsilon_{xx} = \varepsilon_1$, $\varepsilon_{yy} = \varepsilon_2$, $\varepsilon_{zz} = \varepsilon_3$. Write an expression for the axial strain $\varepsilon$ in the arbitrary direction shown in Fig. 2.21.

**Solution**
Choose a coordinate system $(x'\,y'\,z')$ such that the $x'$-axis is along the the arbitrary direction shown in Fig. 2.21.

$$\begin{aligned}\varepsilon = \varepsilon_{x'x'} &= Q_{1k}\,Q_{1l}\,\varepsilon_{kl} \\ &= Q_{11}\,Q_{11}\,\varepsilon_{xx} + Q_{12}\,Q_{12}\,\varepsilon_{yy} + Q_{13}\,Q_{13}\,\varepsilon_{zz}\end{aligned}$$

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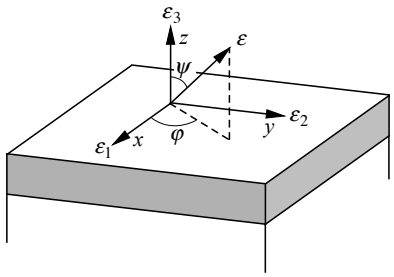



Figure 2.21: A thin film subjected to strain $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ in the $x$, $y$, $z$ directions, respectively.

where $Q_{ik} = (\hat{e}'_i \cdot \hat{e}_k)$.

$$\begin{aligned} Q_{11} &= \sin\psi\cos\varphi \\ Q_{12} &= \sin\psi\sin\varphi \\ Q_{13} &= \cos\psi \end{aligned}$$

Therefore,

$$\varepsilon = (\sin\psi\cos\varphi)^2\,\varepsilon_1 + (\sin\psi\sin\varphi)^2\,\varepsilon_2 + (\cos\psi)^2\,\varepsilon_3$$

**2.10** A solid cube with shear modulus $\mu$ and Poisson's ratio $\nu$ is subjected to a compressive stress along $z$, i.e. $\sigma_{zz} = -p$, but is not allowed to expand in the $x$ direction due to constraints imposed by rigid plates. The solid is allowed to expand freely in the $y$ direction. Obtain the normal stress and strain of the solid in $x$, $y$ and $z$ directions.

**Solution**

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \\ \varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \\ \varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \end{aligned}$$

For this problem, $\varepsilon_{xx} = 0$ and $\sigma_{yy} = 0$. Therefore,

$$\begin{aligned} 0 = \varepsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{zz}\right) \\ \sigma_{xx} &= \nu\,\sigma_{zz} = -\nu\,p \end{aligned}$$

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In summary,

$$\begin{aligned}
\sigma_{xx} &= -\nu\, p \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= -p \\
\epsilon_{xx} &= -\nu\, p \\
\epsilon_{yy} &= -\frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) = \frac{\nu(1+\nu)p}{E} \\
\epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu\,\sigma_{xx}) = -\frac{(1-\nu^2)p}{E}
\end{aligned}$$

**2.11** Repeat the analysis in Problem 2.10 but for a solid constrained (i.e. unable to expand) in both $x$ and $y$ directions.

**Solution**

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right]
\end{aligned}$$

For this problem, $\varepsilon_{xx} = 0$ and $\varepsilon_{yy} = 0$. By symmetry, we have $\sigma_{xx} = \sigma_{yy}$. Therefore,

$$\begin{aligned}
0 = \varepsilon_{xx} &= \frac{1}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{zz}\right] \\
\sigma_{xx} = \sigma_{yy} &= \frac{\nu}{1-\nu}\,\sigma_{zz} = -\frac{\nu}{1-\nu}\,p
\end{aligned}$$

In summary,

$$\begin{aligned}
\sigma_{xx} &= -\frac{\nu}{1-\nu}\,p \\
\sigma_{yy} &= -\frac{\nu}{1-\nu}\,p \\
\sigma_{zz} &= -p \\
\epsilon_{xx} &= 0 \\
\epsilon_{yy} &= 0 \\
\epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - 2\nu\,\sigma_{xx}) = -\frac{p}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)
\end{aligned}$$

**2.12** Estimate the compressive stress that arises in a block of aluminum when the temperature increases from 0°C to 30°C if its volume is not allowed to expand. Use the thermoelastic properties of aluminum given in Table B.1. If the block were allowed to expand freely, it would experience a thermal strain of $\varepsilon^{\mathrm{T}} = \alpha\,\Delta T$ in all three direction with zero stress, where $\alpha$ is the thermal expansion coefficient and $\Delta T$ is the temperature increase.

**Solution**

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$$\varepsilon_{xx}^{\text{tot}} = \varepsilon_{xx}^{\text{el}} + \varepsilon_{xx}^{\text{T}} = \varepsilon_{xx}^{\text{el}} + \alpha\Delta T = 0$$
$$\varepsilon_{yy}^{\text{tot}} = \varepsilon_{yy}^{\text{el}} + \varepsilon_{yy}^{\text{T}} = \varepsilon_{yy}^{\text{el}} + \alpha\Delta T = 0$$
$$\varepsilon_{zz}^{\text{tot}} = \varepsilon_{zz}^{\text{el}} + \varepsilon_{zz}^{\text{T}} = \varepsilon_{zz}^{\text{el}} + \alpha\Delta T = 0$$

Therefore, $\varepsilon_{xx}^{\text{el}} = \varepsilon_{yy}^{\text{el}} = \varepsilon_{zz}^{\text{el}} = -\alpha\Delta T$.

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = B \cdot (\varepsilon_{xx}^{\text{el}} + \varepsilon_{yy}^{\text{el}} + \varepsilon_{zz}^{\text{el}}) = -3B\,\alpha\,\Delta T$$

From Table B.1, we find $B = 75.9$ GPa, $\alpha = 23.1 \times 10^{-6}\,\text{K}^{-1}$ for aluminum. Therefore, for $\Delta T = 30$ K, we have

$$\begin{aligned}\sigma_{xx} = \sigma_{yy} = \sigma_{zz} &= -3B\,\alpha\,\Delta T \\ &= -3 \times 75.9 \times 23.1 \times 10^{-6} \times 30\,\text{GPa} \\ &= 0.158\,\text{GPa} \\ &= 158\,\text{MPa}\end{aligned}$$

**2.13** Consider an infinite elastic solid containing a cylindrical hole of radius $r_0$, which is filled with a gas with pressure $p$. The stress field in the solid is given in Eq. (2.81). Express the coefficient $A$ in terms of the pressure $p$ inside the hole and the hole radius $r_0$. Verify that this stress field satisfies the equilibrium condition, Eq. (2.56).

**Solution**

$$-p = \sigma_{rr}|_{r=r_0} = -\frac{A}{r_0^2}$$
$$A = p\,r_0^2$$

Because $\sigma_{r\theta} = \sigma_{\theta z} = \sigma_{rz} = 0$, and all the remaining stress components are independent of $\theta$ and $z$, it can be easily seen that, $(F_r = F_\theta = F_z = 0)$

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} = 0$$
$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} = 0$$

The only equilibrium condition we need to verify is

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

Because $\sigma_{rr} = -A/r^2$ and $\sigma_{\theta\theta} = A/r^2$,

$$\frac{\partial\sigma_{rr}}{\partial r} = 2\,\frac{A}{r^3}$$
$$\frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = \frac{1}{r}\left(-\frac{A}{r^2} - \frac{A}{r^2}\right) = -2\,\frac{A}{r^3}$$

Therefore,

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

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Because $\sigma_{r\theta} = \sigma_{rz} = 0$, we have

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

**2.14** Given the stress field expression, Eq. (2.81), for an infinite elastic solid containing a pressurized cylindrical hole,

(a) Find the corresponding strain field. Express the strain field in Cartesian ($xyz$) components.

(b) Verify that the strain field satisfies the compatibility condition Eq. (2.67).

(c) Verify that the displacement field $u_r = C/r$, $u_\theta = 0$ is consistent with the strain field, and express the coefficient $C$ in terms of hole radius $r_0$ and pressure $p$.

Assume the solid is a linear isotropic material with with shear modulus $\mu$ and Poisson's ratio $\nu$.

**Solution**

(a) We have the plane strain condition $\varepsilon_{zz} = 0$. To find $\sigma_{zz}$, we use the condition that

$$0 = \varepsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})]$$
$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$$

Given that $\sigma_{rr} = -A/r^2$ and $\sigma_{\theta\theta} = A/r^2$, $A = p\,r_0^2$ (see Problem 2.13), we have

$$\sigma_{zz} = 0$$

So we also have a plane stress condition in the solid (outside the hole).

$$\varepsilon_{rr} = \frac{1}{E}[\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})]$$
$$= -\frac{1+\nu}{E}\frac{A}{r^2} = -\frac{A}{2\mu}\frac{1}{r^2}$$
$$\varepsilon_{\theta\theta} = \frac{1}{E}[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})]$$
$$= \frac{1+\nu}{E}\frac{A}{r^2} = \frac{A}{2\mu}\frac{1}{r^2}$$
$$\varepsilon_{r\theta} = \varepsilon_{rz} = \varepsilon_{\theta z} = 0$$

Similar to the stress transformation rule from cylindrical to Cartesian coordinates derived in Problem 2.6, we have similar expressions for the strain transformation, i.e.

$$Q_{ik} = \begin{bmatrix} Q_{xr} & Q_{x\theta} & Q_{xz} \\ Q_{yr} & Q_{y\theta} & Q_{yz} \\ Q_{zr} & Q_{z\theta} & Q_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$
\begin{aligned}
\varepsilon_{xx} &= Q_{xr}\,Q_{xr}\,\varepsilon_{rr} + Q_{x\theta}\,Q_{x\theta}\,\varepsilon_{\theta\theta} = -(\cos^2\theta - \sin^2\theta)\left(\frac{A}{2\mu\,r^2}\right)\\
\varepsilon_{yy} &= Q_{yr}\,Q_{yr}\,\varepsilon_{rr} + Q_{y\theta}\,Q_{y\theta}\,\varepsilon_{\theta\theta} = (\cos^2\theta - \sin^2\theta)\left(\frac{A}{2\mu\,r^2}\right)\\
\varepsilon_{zz} &= 0\\
\varepsilon_{xy} &= Q_{xr}\,Q_{yr}\,\varepsilon_{rr} + Q_{x\theta}\,Q_{y\theta}\,\varepsilon_{\theta\theta} = -2\sin\theta\cos\theta\left(\frac{A}{2\mu\,r^2}\right)\\
\varepsilon_{yz} &= Q_{yr}\,Q_{zr}\,\varepsilon_{rr} + Q_{y\theta}\,Q_{z\theta}\,\varepsilon_{\theta\theta} = 0\\
\varepsilon_{xz} &= Q_{xr}\,Q_{zr}\,\varepsilon_{rr} + Q_{x\theta}\,Q_{z\theta}\,\varepsilon_{\theta\theta} = 0
\end{aligned}
$$

The results can be simplified as: $\sigma_{xx} = -A\cos(2\theta)/r^2$, $\sigma_{yy} = A\cos(2\theta)/r^2$, $\sigma_{xy} = -A\sin(2\theta)/r^2$. They can be written in terms of $x$, $y$, $z$ as

$$
\begin{aligned}
\varepsilon_{xx} &= -\frac{A}{2\mu}\frac{x^2-y^2}{(x^2+y^2)^2}\\
\varepsilon_{yy} &= \frac{A}{2\mu}\frac{x^2-y^2}{(x^2+y^2)^2}\\
\varepsilon_{xy} &= -\frac{A}{2\mu}\frac{2xy}{(x^2+y^2)^2}
\end{aligned}
$$

(b) The compatibility condition to be verified is

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$$

$$
\begin{aligned}
\varepsilon_{xx,yy} &= \frac{\partial^2}{\partial y^2}\varepsilon_{xx} = \frac{A}{2\mu}\frac{6(x^4-6xy+y^4)}{(x^2+y^2)^4}\\
\varepsilon_{yy,xx} &= \frac{\partial^2}{\partial x^2}\varepsilon_{yy} = \frac{A}{2\mu}\frac{6(x^4-6xy+y^4)}{(x^2+y^2)^4}\\
\varepsilon_{xy,xy} &= \frac{\partial^2}{\partial x\partial y}\varepsilon_{xy} = \frac{A}{2\mu}\frac{6(x^4-6xy+y^4)}{(x^2+y^2)^4}
\end{aligned}
$$

Therefore,

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$$

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(c) Suppose $u_r = C/r$ and $u_\theta = u_z = 0$, then

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} = -\frac{C}{r^2} \\
\varepsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{C}{r^2} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} = 0 \\
\varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) = 0 \\
\varepsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0 \\
\varepsilon_{\theta z} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) = 0
\end{aligned}$$

These are consistent with the displacement fields in (a) as long as

$$C = \frac{A}{2\,\mu} = \frac{p\,r_0^2}{2\,\mu}$$

**2.15** Given the stress and strain expressions in Problems 2.13 and 2.14 for an infinite solid containing a pressurized cylindrical hole, express the total elastic energy per unit length along the hole in terms of the pressure $p$ and hole radius $r_0$ using the following two approaches and show that they agree with each other.

(a) Integrate the strain energy density $\Delta w$ over the entire volume of the solid.

(b) Integrate the work done on the interior surface of the cylindrical hole as the pressure gradually increases from 0 to $p$.

**Solution**

(a)

$$\begin{aligned}
\Delta w &= \frac{1}{2}\sigma_{rr}\varepsilon_{rr} + \frac{1}{2}\sigma_{\theta\theta}\varepsilon_{\theta\theta} = \frac{1}{2}\,\frac{A^2}{2\mu\,r^4} + \frac{1}{2}\,\frac{A^2}{2\mu\,r^4} \\
&= \frac{A^2}{2\mu\,r^4} \\
E &= \int_{r_0}^{\infty} r dr \int_0^{2\pi} d\theta\,\Delta w = \int_{r_0}^{\infty} dr\,2\pi\,r\,\frac{A^2}{2\mu\,r^4} \\
&= \frac{\pi\,A^2}{2\,\mu\,r_0^2} = \frac{\pi\,(p\,r_0^2)^2}{2\,\mu\,r_0^2} = \frac{\pi\,p^2\,r_0^2}{2\,\mu}
\end{aligned}$$

(b)

$$E = \frac{1}{2}\,p\,(2\pi\,r_0)\,u_r|_{r=r_0}$$

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From Problem 2.14, we have $u_r = C/r$ with $C = p\,r_0^2/(2\mu)$, so that

$$u_r|_{r=r_0} = \frac{p\,r_0^2}{2\,\mu}\,\frac{1}{r_0}$$

$$E = \frac{1}{2}\,p\,(2\pi\,r_0)\,\left(\frac{p\,r_0^2}{2\,\mu}\,\frac{1}{r_0}\right) = \frac{\pi\,p^2\,r_0^2}{2\,\mu}$$

consistent with the result in (a).

**2.16** Find the expressions for the displacement field for Kelvin's problem, in which a unit force in the $z$ direction is applied at the origin in an infinite elastic medium. Express the displacement field in cylindrical coordinates. Which strain and stress components are non-zero in cylindrical coordinates?

**Solution**

In Cartesian coordinates, the displacement field caused by a point force in the $z$-direction is,

$$u_{\rm x} = \frac{1}{16\pi\mu(1-\nu)}\,\frac{x\,z}{(x^2+y^2+z^2)^{3/2}}$$

$$u_{\rm y} = \frac{1}{16\pi\mu(1-\nu)}\,\frac{y\,z}{(x^2+y^2+z^2)^{3/2}}$$

$$u_{\rm z} = \frac{1}{16\pi\mu(1-\nu)}\left[\frac{3-4\nu}{(x^2+y^2+z^2)^{1/2}} + \frac{z^2}{(x^2+y^2+z^2)^{3/2}}\right]$$

In cylindrical coordinates $(r\,\theta\,z)$, where $r = \sqrt{x^2+y^2}$, the displacement field is,

$$u_r = \frac{1}{16\pi\mu(1-\nu)}\,\frac{r\,z}{(r^2+z^2)^{3/2}}$$

$$u_\theta = 0$$

$$u_z = \frac{1}{16\pi\mu(1-\nu)}\left[\frac{3-4\nu}{(r^2+z^2)^{1/2}} + \frac{z^2}{(r^2+z^2)^{3/2}}\right]$$

In cylindrical coordinates, the strain field is related to the displacement field through,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r},\quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r},\quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right)$$

Given that $u_r$ and $u_z$ are independent of $\theta$ and $u_\theta = 0$, we have $\varepsilon_{r\theta} = \varepsilon_{\theta z} = 0$. The non-zero strain components are: $\varepsilon_{rr}$, $\varepsilon_{\theta\theta}$, $\varepsilon_{zz}$, and $\varepsilon_{rz}$.

From generalized Hooke's law, we have $\sigma_{r\theta} = \sigma_{\theta z} = 0$. The non-zero stress components are: $\sigma_{rr}$, $\sigma_{\theta\theta}$, $\sigma_{zz}$, and $\sigma_{rz}$.

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