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CHAPTER 2
Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

SECTION 2.1 Sets

2. There are of course an infinite number of correct answers.

a) $\{3n \mid n = 0, 1, 2, 3, 4\}$ or $\{x \mid x$ is a multiple of $3 \wedge 0 \leq x \leq 12\}$.

b) $\{x \mid -3 \leq x \leq 3\}$, where we are assuming that the domain (universe of discourse) is the set of integers.

c) $\{x \mid x$ is a letter of the word *monopoly* other than l or $y\}$.

4. **a)** a

b) It is empty because no number is both greater than or equal to a and at the same time less than a.

c) It is empty because no number is both greater than a and at the same time less than or equal to a.

d) It is empty because no number is both greater than a and at the same time less than a.

e) It is empty, because if x were in this set, then $a < x < b$, which would imply that $a < b$, but we are given that $a > b$.

f) It is empty, because if x were in this set, then $a \leq x \leq b$, which would imply that $a \leq b$, but we are given that $a > b$.

6. Recall that one set is a subset of another set if every element of the first set is also an element of the second.

a) The second condition imposes an extra requirement, so clearly the second set is a subset of the first, but not vice versa.

b) Again the second condition imposes an extra requirement, so the second set is a subset of the first, but not vice versa.

c) There could well be students studying discrete mathematics but not data structures (for example, pure math majors) and students studying data structure but not discrete mathematics (at least not this semester—one could argue that the knowing the latter is necessary to really understand the former!), so neither set is a subset of the other.

8. Each of the sets is a subset of itself. Aside from that, the only relations are $B \subseteq A$, $C \subseteq A$, and $C \subseteq D$.

10. **a)** Since the set contains only integers and $\{2\}$ is a set, not an integer, $\{2\}$ is not an element.

b) Since the set contains only integers and $\{2\}$ is a set, not an integer, $\{2\}$ is not an element.

c) The set has two elements. One of them is patently $\{2\}$.

d) The set has two elements. One of them is patently $\{2\}$.

e) The set has two elements. One of them is patently $\{2\}$.

f) The set has only one element, $\{\{2\}\}$; since this is not the same as $\{2\}$ (the former is a set containing a set, whereas the latter is a set containing a number), $\{2\}$ is not an element of $\{\{\{2\}\}\}$.

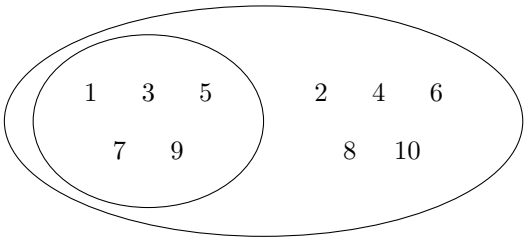
12. **a)** true **b)** true **c)** false—see part (a) **d)** true

e) true—the one element in the set on the left is an element of the set on the right, and the sets are not equal

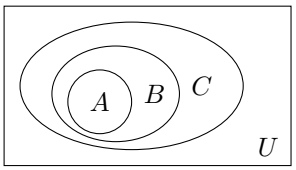
f) true—similar to part (e) **g)** false—the two sets are equal

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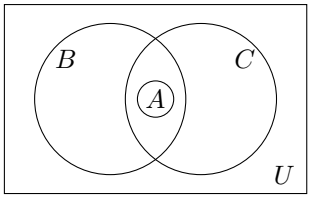
14. The numbers 1, 3, 5, 7, and 9 form a subset of the set of all ten positive integers under discussion, as shown here.



16. We put the subsets inside the supersets. Thus the answer is as shown.



18. We allow B and C to overlap, because we are told nothing about their relationship. The set A must be a subset of each of them, and that forces it to be positioned as shown. We cannot actually show the properness of the subset relationships in the diagram, because we don't know where the elements in B and C that are not in A are located—there might be only one (which is in both B and C), or they might be located in portions of B and/or C outside the other. Thus the answer is as shown, but with the added condition that there must be at least one element of B not in A and one element of C not in A.



20. Since the empty set is a subset of every set, we just need to take a set B that contains $\emptyset$ as an element. Thus we can let $A = \emptyset$ and $B = \{\emptyset\}$ as the simplest example.

22. The cardinality of a set is the number of elements it has.

a) The empty set has no elements, so its cardinality is 0.

b) This set has one element (the empty set), so its cardinality is 1.

c) This set has two elements, so its cardinality is 2.

d) This set has three elements, so its cardinality is 3.

24. The union of all the sets in the power set of a set X must be exactly X. In other words, we can recover X from its power set, uniquely. Therefore the answer is yes.

26. **a)** The power set of every set includes at least the empty set, so the power set cannot be empty. Thus $\emptyset$ is not the power set of any set.

b) This is the power set of $\{a\}$.

c) This set has three elements. Since 3 is not a power of 2, this set cannot be the power set of any set.

d) This is the power set of $\{a, b\}$.

28. We need to show that every element of $A \times B$ is also an element of $C \times D$. By definition, a typical element of $A \times B$ is a pair (a, b) where $a \in A$ and $b \in B$. Because $A \subseteq C$, we know that $a \in C$; similarly, $b \in D$. Therefore $(a, b) \in C \times D$.

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30. By definition it is the set of all ordered pairs (c, p) such that c is a course and p is a professor. The elements of this set are the possible teaching assignments for the mathematics department.

32. We can conclude that $A = \emptyset$ or $B = \emptyset$. To prove this, suppose that neither A nor B were empty. Then there would be elements $a \in A$ and $b \in B$. This would give at last one element, namely (a, b), in $A \times B$, so $A \times B$ would not be the empty set. This contradiction shows that either A or B (or both, it goes without saying) is empty.

34. In each case the answer is a set of 3-tuples.

a) $\{(a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1), (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1), (c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1)\}$

b) $\{(0, x, a), (0, x, b), (0, x, c), (0, y, a), (0, y, b), (0, y, c), (1, x, a), (1, x, b), (1, x, c), (1, y, a), (1, y, b), (1, y, c)\}$

c) $\{(0, a, x), (0, a, y), (0, b, x), (0, b, y), (0, c, x), (0, c, y), (1, a, x), (1, a, y), (1, b, x), (1, b, y), (1, c, x), (1, c, y)\}$

d) $\{(x, x, x), (x, x, y), (x, y, x), (x, y, y), (y, x, x), (y, x, y), (y, y, x), (y, y, y)\}$

36. Recall that A^3 consists of all the ordered triples (x, y, z) of elements of A.

a) $\{(a, a, a)\}$ **b)** $\{(0, 0, 0), (0, 0, a), (0, a, 0), (0, a, a), (a, 0, 0), (a, 0, a), (a, a, 0), (a, a, a)\}$

38. The set $A \times B \times C$ consists of ordered triples (a, b, c) with $a \in A$, $b \in B$, and $c \in C$. There are m choices for the first coordinate. For each of these, there n choices for the second coordinate, giving us mn choices for the first two coordinates. For each of these, there p choices for the third coordinate, giving us mnp choices in all. Therefore $A \times B \times C$ has mnp elements. This is an application of the product rule (see Chapter 6).

40. Suppose $A \neq B$ and neither A nor B is empty. We must prove that $A \times B \neq B \times A$. Since $A \neq B$, either we can find an element x that is in A but not B, or vice versa. The two cases are similar, so without loss of generality, let us assume that x is in A but not B. Also, since B is not empty, there is some element $y \in B$. Then (x, y) is in $A \times B$ by definition, but it is not in $B \times A$ since $x \notin B$. Therefore $A \times B \neq B \times A$.

42. The only difference between $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ is parentheses, so for all practical purposes one can think of them as essentially the same thing. By Definition 8, the elements of $(A \times B) \times (C \times D)$ consist of ordered pairs (x, y), where $x \in A \times B$ and $y \in C \times D$, so the typical element of $(A \times B) \times (C \times D)$ looks like $((a, b), (c, d))$. By Definition 9, the elements of $A \times (B \times C) \times D$ consist of 3-tuples (a, x, d), where $a \in A$, $d \in D$, and $x \in B \times C$, so the typical element of $A \times (B \times C) \times D$ looks like $(a, (b, c), d)$. The structures $((a, b), (c, d))$ and $(a, (b, c), d)$ are different, even if they convey exactly the same information (the first is a pair, and the second is a 3-tuple). To be more precise, there is a natural one-to-one correspondence between $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ given by $((a, b), (c, d)) \leftrightarrow (a, (b, c), d)$.

44. This is true. To show that $B = C$, it is enough to show that every element of B is also an element of C and vice versa. By symmetry, it suffices to prove only the first of these. Let $b \in B$, and let a be any element in the nonempty set A. Then $(a, b) \in A \times B$, and because $A \times B = A \times C$, this means that $(a, b) \in A \times C$. From the definition of $A \times C$, it follows that $b \in C$.

46. **a)** There is a real number whose cube is -1. This is true, since $x = -1$ is a solution.

b) There is an integer such that the number obtained by adding 1 to it is greater than the integer. This is true—in fact, every integer satisfies this statement.

c) For every integer, the number obtained by subtracting 1 is again an integer. This is true.

d) The square of every integer is an integer. This is true.

48. In each case we want the set of all values of x in the domain (the set of integers) that satisfy the given equation or inequality.

a) It is exactly the positive integers that satisfy this inequality. Therefore the truth set is $\{x \in \mathbf{Z} \mid x^3 \geq 1\} = \{x \in \mathbf{Z} \mid x \geq 1\} = \{1, 2, 3, \ldots\}$.

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b) The square roots of 2 are not integers, so the truth set is the empty set, $\emptyset$.

c) Negative integers certainly satisfy this inequality, as do all positive integers greater than 1. However, $0 \not< 0^2$ and $1 \not< 1^2$. Thus the truth set is $\{x \in \mathbf{Z} \mid x < x^2\} = \{x \in \mathbf{Z} \mid x \neq 0 \wedge x \neq 1\} = \{\ldots, -3, -2, -1, 2, 3, \ldots\}$.

50. a) If $S \in S$, then by the defining condition for S we conclude that $S \notin S$, a contradiction.

b) If $S \notin S$, then by the defining condition for S we conclude that it is not the case that $S \notin S$ (otherwise S would be an element of S), again a contradiction.

SECTION 2.2 Set Operations

2. a) $A \cap B$ **b)** $A \cap \overline{B}$, which is the same as $A - B$ **c)** $A \cup B$ **d)** $\overline{A} \cup \overline{B}$

4. Note that $A \subseteq B$.

a) $\{a, b, c, d, e, f, g, h\} = B$ **b)** $\{a, b, c, d, e\} = A$

c) There are no elements in A that are not in B, so the answer is $\emptyset$. **d)** $\{f, g, h\}$

6. a) $A \cup \emptyset = \{\, x \mid x \in A \vee x \in \emptyset \,\} = \{\, x \mid x \in A \vee \mathbf{F} \,\} = \{\, x \mid x \in A \,\} = A$

b) $A \cap U = \{\, x \mid x \in A \wedge x \in U \,\} = \{\, x \mid x \in A \wedge \mathbf{T} \,\} = \{\, x \mid x \in A \,\} = A$

8. a) $A \cup A = \{\, x \mid x \in A \vee x \in A \,\} = \{\, x \mid x \in A \,\} = A$

b) $A \cap A = \{\, x \mid x \in A \wedge x \in A \,\} = \{\, x \mid x \in A \,\} = A$

10. a) $A - \emptyset = \{\, x \mid x \in A \wedge x \notin \emptyset \,\} = \{\, x \mid x \in A \wedge \mathbf{T} \,\} = \{\, x \mid x \in A \,\} = A$

b) $\emptyset - A = \{\, x \mid x \in \emptyset \wedge x \notin A \,\} = \{\, x \mid \mathbf{F} \wedge x \notin A \,\} = \{\, x \mid \mathbf{F} \,\} = \emptyset$

12. We will show that these two sets are equal by showing that each is a subset of the other. Suppose $x \in A \cup (A \cap B)$. Then $x \in A$ or $x \in A \cap B$ by the definition of union. In the former case, we have $x \in A$, and in the latter case we have $x \in A$ and $x \in B$ by the definition of intersection; thus in any event, $x \in A$, so we have proved that the left-hand side is a subset of the right-hand side. Conversely, let $x \in A$. Then by the definition of union, $x \in A \cup (A \cap B)$ as well. Thus we have shown that the right-hand side is a subset of the left-hand side.

14. Since $A = (A - B) \cup (A \cap B)$, we conclude that $A = \{1, 5, 7, 8\} \cup \{3, 6, 9\} = \{1, 3, 5, 6, 7, 8, 9\}$. Similarly $B = (B - A) \cup (A \cap B) = \{2, 10\} \cup \{3, 6, 9\} = \{2, 3, 6, 9, 10\}$.

16. a) If x is in $A \cap B$, then perforce it is in A (by definition of intersection).

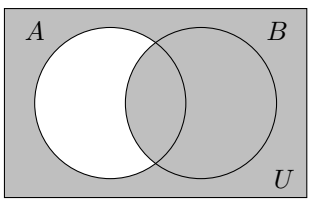
b) If x is in A, then perforce it is in $A \cup B$ (by definition of union).

c) If x is in $A - B$, then perforce it is in A (by definition of difference).

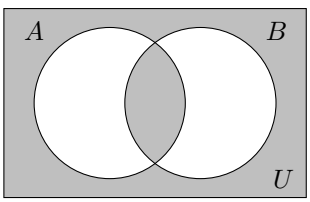
d) If $x \in A$ then $x \notin B - A$. Therefore there can be no elements in $A \cap (B - A)$, so $A \cap (B - A) = \emptyset$.

e) The left-hand side consists precisely of those things that are either elements of A or else elements of B but not A, in other words, things that are elements of either A or B (or, of course, both). This is precisely the definition of the right-hand side.

18. a)



b)



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20. **a)** Suppose that $x \in A \cup B$. Then either $x \in A$ or $x \in B$. In either case, certainly $x \in A \cup B \cup C$. This establishes the desired inclusion.

b) Suppose that $x \in A \cap B \cap C$. Then x is in all three of these sets. In particular, it is in both A and B and therefore in $A \cap B$, as desired.

c) Suppose that $x \in (A - B) - C$. Then x is in $A - B$ but not in C. Since $x \in A - B$, we know that $x \in A$ (we also know that $x \notin B$, but that won't be used here). Since we have established that $x \in A$ but $x \notin C$, we have proved that $x \in A - C$.

d) To show that the set given on the left-hand side is empty, it suffices to assume that x is some element in that set and derive a contradiction, thereby showing that no such x exists. So suppose that $x \in (A-C) \cap (C-B)$. Then $x \in A - C$ and $x \in C - B$. The first of these statements implies by definition that $x \notin C$, while the second implies that $x \in C$. This is impossible, so our proof by contradiction is complete.

e) To establish the equality, we need to prove inclusion in both directions. To prove that $(B-A) \cup (C-A) \subseteq (B \cup C) - A$, suppose that $x \in (B-A) \cup (C-A)$. Then either $x \in (B-A)$ or $x \in (C-A)$. Without loss of generality, assume the former (the proof in the latter case is exactly parallel.) Then $x \in B$ and $x \notin A$. From the first of these assertions, it follows that $x \in B \cup C$. Thus we can conclude that $x \in (B \cup C) - A$, as desired. For the converse, that is, to show that $(B \cup C) - A \subseteq (B - A) \cup (C - A)$, suppose that $x \in (B \cup C) - A$. This means that $x \in (B \cup C)$ and $x \notin A$. The first of these assertions tells us that either $x \in B$ or $x \in C$. Thus either $x \in B - A$ or $x \in C - A$. In either case, $x \in (B - A) \cup (C - A)$. (An alternative proof could be given by using Venn diagrams, showing that both sides represent the same region.)

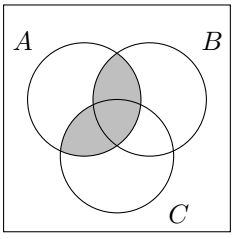
22. **a)** It is always the case that $B \subseteq A \cup B$, so it remains to show that $A \cup B \subseteq B$. But this is clear because if $x \in A \cup B$, then either $x \in A$, in which case $x \in B$ (because we are given $A \subseteq B$) or $x \in B$; in either case $x \in B$.

b) It is always the case that $A \cap B \subseteq A$, so it remains to show that $A \subseteq A \cap B$. But this is clear because if $x \in A$, then $x \in B$ as well (because we are given $A \subseteq B$), so $x \in A \cap B$.

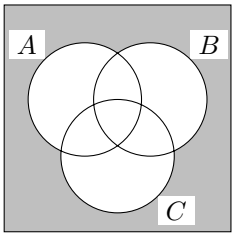
24. First we show that every element of the left-hand side must be in the right-hand side as well. If $x \in A \cap (B \cap C)$, then x must be in A and also in $B \cap C$. Hence x must be in A and also in B and in C. Since x is in both A and B, we conclude that $x \in A \cap B$. This, together with the fact that $x \in C$ tells us that $x \in (A \cap B) \cap C$, as desired. The argument in the other direction (if $x \in (A \cap B) \cap C$ then x must be in $A \cap (B \cap C)$) is nearly identical.

26. First suppose x is in the left-hand side. Then x must be in A but in neither B nor C. Thus $x \in A - C$, but $x \notin B - C$, so x is in the right-hand side. Next suppose that x is in the right-hand side. Thus x must be in $A - C$ and not in $B - C$. The first of these implies that $x \in A$ and $x \notin C$. But now it must also be the case that $x \notin B$, since otherwise we would have $x \in B - C$. Thus we have shown that x is in A but in neither B nor C, which implies that x is in the left-hand side.

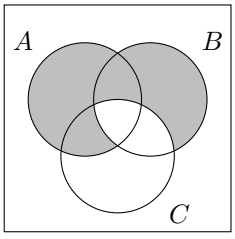
28. The set is shaded in each case.



(a)



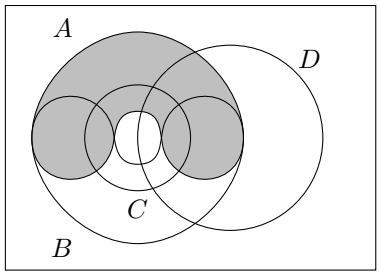
(b)



(c)

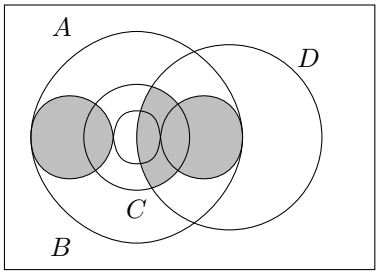
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30. Here is a Venn diagram that can be used for four sets. Notice that sets A and B are not convex in this picture. We have shaded set A. Notice that each of the 16 different combinations are represented by a region.

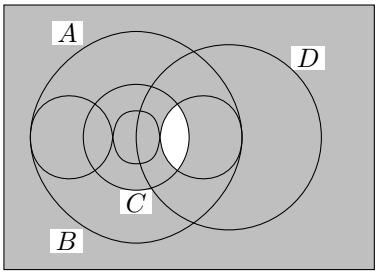


We can now shade in the appropriate regions for each of the expressions in this exercise.

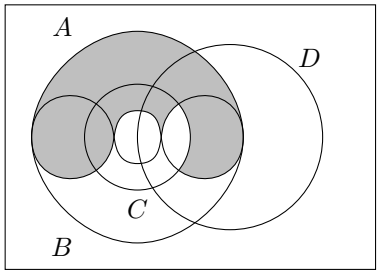
a)



b)



c)



32. **a)** We cannot conclude that $A = B$. For instance, if A and B are both subsets of C, then this equation will always hold, and A need not equal B.

b) We cannot conclude that $A = B$; let $C = \emptyset$, for example.

c) By putting the two conditions together, we *can* now conclude that $A = B$. By symmetry, it suffices to prove that $A \subseteq B$. Suppose that $x \in A$. There are two cases. If $x \in C$, then $x \in A \cap C = B \cap C$, which forces $x \in B$. On the other hand, if $x \notin C$, then because $x \in A \cup C = B \cup C$, we must have $x \in B$.

34. By the given identity and the commutative, associative, and absorption laws,

$$(A - B) \cap (B - C) \cap (A - C) = (A \cap \overline{B}) \cap (B \cap \overline{C}) \cap (A \cap \overline{C}) = A \cap (B \cap \overline{B}) \cap \overline{C}.$$

By the complement law, the term in parentheses is $\emptyset$, so by the domination law, the expression equals $\emptyset$.

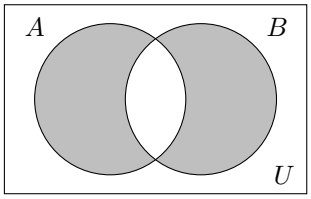
36. **a)** Let $(x, y) \in A \times (B \cup C)$, which means that $x \in A$ and y is an element of either B or C. Thus $(x, y) \in A \times B$ or $(x, y) \in A \times C$, so by the definition of union, $(x, y) \in (A \times B) \cup (A \times C)$. Conversely let $(x, y) \in (A \times B) \cup (A \times C)$. Then either $(x, y) \in A \times B$ or $(x, y) \in A \times C$. Thus $x \in A$, and either $y \in B$ or $y \in C$. This implies that $y \in B \cup C$, so $(x, y) \in A \times (B \cup C)$.

b) Let $(x, y) \in A \times (B \cap C)$, which means that $x \in A$ and y is an element of both B and C. Thus $(x, y) \in A \times B$ and $(x, y) \in A \times C$, so by the definition of intersection, $(x, y) \in (A \times B) \cap (A \times C)$. Conversely let $(x, y) \in (A \times B) \cap (A \times C)$. Then both $(x, y) \in A \times B$ and $(x, y) \in A \times C$. Thus $x \in A$, and both $y \in B$ and $y \in C$. This implies that $y \in B \cap C$, so $(x, y) \in A \times (B \cap C)$.

38. This is the set of elements in exactly one of these sets, namely $\{2, 5\}$.

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40. The figure is as shown; we shade that portion of A that is not in B and that portion of B that is not in A.



42. There are precisely two ways that an item can be in either A or B but not both. It can be in A but not B (which is equivalent to saying that it is in $A-B$), or it can be in B but not A (which is equivalent to saying that it is in $B-A$). Thus an element is in $A \oplus B$ if and only if it is in $(A-B)\cup(B-A)$.

44. **a)** This is clear from the symmetry (between A and B) in the definition of symmetric difference.

b) We prove two things. To show that $A \subseteq (A \oplus B) \oplus B$, suppose $x \in A$. If $x \in B$, then $x \notin A \oplus B$, so x is an element of the right-hand side. On the other hand if $x \notin B$, then $x \in A \oplus B$, so again x is in the right-hand side. Conversely, suppose x is an element of the right-hand side. There are two cases. If $x \notin B$, then necessarily $x \in A \oplus B$, whence $x \in A$. If $x \in B$, then necessarily $x \notin A \oplus B$, and the only way for that to happen (since $x \in B$) is for x to be in A.

46. This is an identity; each side consists of those things that are in an odd number of the sets A, B, and C.

48. This is an identity; each side consists of those things that are in an odd number of the sets A, B, C, and D.

50. A finite set is a set with k elements for some natural number k. Suppose that A has n elements and B has m elements. Then the number of elements in $A \cup B$ is at most $n+m$ (it might be less because $A \cap B$ might be nonempty). Therefore by definition, $A \cup B$ is finite.

52. To count the elements of $A \cup B \cup C$ we proceed as follows. First we count the elements in each of the sets and add. This certainly gives us all the elements in the union, but we have overcounted. Each element in $A \cap B$, $A \cap C$, and $B \cap C$ has been counted twice. Therefore we subtract the cardinalities of these intersections to make up for the overcount. Finally, we have compensated a bit too much, since the elements of $A \cap B \cap C$ have now been counted three times and subtracted three times. We adjust by adding back the cardinality of $A \cap B \cap C$.

54. We note that these sets are increasing, that is, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. Therefore, the union of any collection of these sets is just the one with the largest subscript, and the intersection is just the one with the smallest subscript.

a) $A_n = \{\ldots, -2, -1, 0, 1, \ldots, n\}$ **b)** $A_1 = \{\ldots, -2, -1, 0, 1\}$

56. **a)** As i increases, the sets get smaller: $\cdots \subset A_3 \subset A_2 \subset A_1$. All the sets are subsets of A_1, which is the set of positive integers, $\mathbf{Z}^+$. It follows that $\bigcup_{i=1}^{\infty} A_i = \mathbf{Z}^+$. Every positive integer is excluded from at least one of the sets (in fact from infinitely many), so $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

b) All the sets are subsets of the set of natural numbers $\mathbf{N}$ (the nonnegative integers). The number 0 is in each of the sets, and every positive integer is in exactly one of the sets, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$ and $\bigcap_{i=1}^{\infty} A_i = \{0\}$.

c) As i increases, the sets get larger: $A_1 \subset A_2 \subset A_3 \cdots$. All the sets are subsets of the set of positive real numbers $\mathbf{R}^+$, and every positive real number is included eventually, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{R}^+$. Because A_1 is a subset of each of the others, $\bigcap_{i=1}^{\infty} A_i = A_1 = (0,1)$ (the interval of all real numbers between 0 and 1, exclusive).

d) This time, as in part (a), the sets are getting smaller as i increases: $\cdots \subset A_3 \subset A_2 \subset A_1$. Because A_1 includes all the others, $\bigcup_{i=1}^{\infty} A_1 = (1, \infty)$ (all real numbers greater than 1). Every number eventually gets excluded as i increases, so $\bigcap_{i=1}^{\infty} A_i = \emptyset$. Notice that ∞ is not a real number, so we cannot write $\bigcap_{i=1}^{\infty} A_i = \{\infty\}$.

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58. **a)** 00 1110 0000 **b)** 10 1001 0001 **c)** 01 1100 1110

60. **a)** No elements are included, so this is the empty set.

b) All elements are included, so this is the universal set.

62. The bit string for the symmetric difference is obtained by taking the bitwise exclusive *OR* of the two bit strings for the two sets, since we want to include those elements that are in one set or the other but not both.

64. We can take the bitwise *OR* (for union) or *AND* (for intersection) of all the bit strings for these sets.

66. The successor set has one more element than the original set, namely the original set itself. Therefore the answer is $n+1$.

68. Because these are ordinary sets and not multisets, every set in this exercise is just $\{a\}$. Thus (a), (b), (c), and (d) are all true by the idempotent laws, and (e) is false because $\{a\} - \{a\} = \emptyset$.

70. **a)** If the departments share the equipment, then the maximum number of each type is all that is required, so we want to take the union of the multisets, $A \cup B$.

b) Both departments will use the minimum number of each type, so we want to take the intersection of the multisets, $A \cap B$.

c) This will be the difference $B - A$ of the multisets.

d) If no sharing is allowed, then the university needs to purchase a quantity of each type of equipment that is the sum of the quantities used by the departments; this is the sum of the multisets, $A + B$.

72. **a)** This follows immediately from the idempotent laws: $A \cap A = A \cup A = A$.

b) This follows immediately from the commutative laws for intersection and union.

c) The "if" part is part (a) of this exercise. Conversely, if $A \neq B$, then $A \cup B$ must be a proper superset of A, so $|A \cup B| > |A|$, whereas $|A \cap B| \leq |A|$, so $J(A,B) < 1$.

d) These follow immediately from the fact that $A \cap B \subseteq A \cup B$, so $|A \cap B| \leq |A \cup B|$.

e) We begin with some preliminary observations and reminders. First, we assume throughout that all sets are finite. Second, note that for real numbers x, y, and z, with $0 < x \leq y$ and $z \geq 0$, $\frac{x}{y} \leq \frac{x+z}{y+z}$ and $\frac{x}{y+z} \leq \frac{x}{y}$.

Third, recall that $A \oplus B$ denotes the symmetric difference of sets, which consists of the elements in A or in B, but not both. By Exercises 41 and 42, $A \oplus B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A)$. Note that the first equality implies that $|A \oplus B| = |A \cup B| - |A \cap B|$.

Fourth, by definition, $d_J(A,B) = 1 - \frac{|A \cap B|}{|A \cup B|}$. By writing the difference as a single fraction and applying the formula for the number of elements in the symmetric difference, we have that $d_J(A,B) = \frac{|A \oplus B|}{|A \cup B|}$.

Fifth, we claim that $B \subseteq (A \oplus B) \cup A$. For any $b \in B$, consider two cases: either $b \in A$ or $b \notin A$. In the second case, we have $b \in B - A \subseteq A \oplus B$. So in both cases, $b \in (A \oplus B) \cup A$. Consequently, $|B| \leq |A \oplus B| + |A|$.

The final preliminary observation is that, for any finite sets A, B, and C, $|A \oplus C| \leq |A \oplus B| + |B \oplus C|$. To see this, we will show that $A \oplus C \subseteq (A \oplus B) \cup (B \oplus C)$. Assume that $x \in A \oplus C$. Without loss of generality, we may assume that $x \in A - C$. Consider the two cases that $x \in B$ or $x \notin B$. In the case that $x \in B$, since $x \notin C$, we have $x \in B - C$ and thus $x \in B \oplus C$. In the other case, that $x \notin B$, then since $x \in A$, we have $x \in A - B$ and $x \in A \oplus B$. So, in either case $x \in (A \oplus B) \cup (B \oplus C)$, and the set inclusion and inequality follow. Furthermore, note that the inequality $|A \oplus C| \leq |A \oplus B| + |B \oplus C|$ implies that $|A \oplus B| + |B \oplus C| - |A \oplus C| \geq 0$. By changing variables, this also yields $|B \oplus C| + |C \oplus \emptyset| - |B \oplus \emptyset| \geq 0$, which is $|B \oplus C| + |C| - |B| \geq 0$. And similarly, $|A \oplus B| + |A| - |B| \geq 0$.

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Now we work to prove the main claim, that for sets A, B, and C, $d_J(A,C) \le d_J(A,B) + d_J(B,C)$.

$$
\begin{aligned}
d_J(A,C) &= \frac{|A \oplus C|}{|A \cup C|} \\
&= \frac{2|A \oplus C|}{2|A \cup C|} \\
&= \frac{2|A \oplus C|}{|A \cup C| + |A \cup C|} \\
&= \frac{2|A \oplus C|}{|A \cup C| + |A| + |C| - |A \cap C|} && \text{by the inclusion-exclusion identity} \\
&= \frac{2|A \oplus C|}{|A \oplus C| + |A| + |C|} && \text{by the third observation above} \\
&= \frac{|A \oplus C| + |A \oplus C|}{|A \oplus C| + |A| + |C|} \\
&\le \frac{|A \oplus B| + |B \oplus C| + |A \oplus C|}{|A \oplus C| + |A| + |C|} && \text{by the final observation} \\
&\le \frac{|A \oplus B| + |B \oplus C| + |A \oplus C| + (|A \oplus B| + |B \oplus C| - |A \oplus C|)}{|A \oplus C| + |A| + |C| + (|A \oplus B| + |B \oplus C| - |A \oplus C|)} && \text{by the second and final observations} \\
&= \frac{2|A \oplus B| + 2|B \oplus C|}{|A| + |C| + |A \oplus B| + |B \oplus C|} \\
&= \frac{2|A \oplus B|}{|A| + |C| + |A \oplus B| + |B \oplus C|} + \frac{2|B \oplus C|}{|A| + |C| + |A \oplus B| + |B \oplus C|} \\
&= \frac{2|A \oplus B|}{|A| + |B| + |A \oplus B| + |B \oplus C| + |C| - |B|} \\
&\quad + \frac{2|B \oplus C|}{|B| + |C| + |B \oplus C| + |A \oplus B| + |A| - |B|} && \text{adding } |B| - |B| \text{ to both denominators and rearranging} \\
&= \frac{2|A \oplus B|}{2|A \cup B| + |B \oplus C| + |C| - |B|} \\
&\quad + \frac{2|B \oplus C|}{2|B \cup C| + |A \oplus B| + |A| - |B|} && \text{since } |A \oplus B| = |A \cup B| - |A \cap B| \text{ and } |A| + |B| - |A \cap B| = |A \cup B| \text{ and similarly for } B \oplus C \\
&\le \frac{2|A \oplus B|}{2|A \cup B|} + \frac{2|B \oplus C|}{2|B \cup C|} && \text{by the final observation and since } \tfrac{x}{y+z} \le \tfrac{x}{y} \\
&= d_J(A,B) + d_J(B,C)
\end{aligned}
$$

A more general approach to proving this result is via the Steinhaus transform, which is described on the webpage `https://mathoverflow.net/questions/18084/is-the-jaccard-distance-a-distance`.

74. Taking the maximum for each person, we have $S \cup T = \{0.6 \text{ Alice}, 0.9 \text{ Brian}, 0.4 \text{ Fred}, 0.9 \text{ Oscar}, 0.7 \text{ Rita}\}$.

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SECTION 2.3 Functions

2. **a)** This is not a function because the rule is not well-defined. We do not know whether $f(3) = 3$ or $f(3) = -3$. For a function, it cannot be both at the same time.

b) This is a function. For all integers n, $\sqrt{n^2+1}$ is a well-defined real number.

c) This is not a function with domain $\mathbf{Z}$, since for $n = 2$ (and also for $n = -2$) the value of $f(n)$ is not defined by the given rule. In other words, $f(2)$ and $f(-2)$ are not specified since division by 0 makes no sense.

4. **a)** The domain is the set of nonnegative integers, and the range is the set of digits (0 through 9).

b) The domain is the set of positive integers, and the range is the set of integers greater than 1.

c) The domain is the set of all bit strings, and the range is the set of nonnegative integers.

d) The domain is the set of all bit strings, and the range is the set of nonnegative integers (a bit string can have length 0).

6. **a)** The domain is $\mathbf{Z}^+ \times \mathbf{Z}^+$ and the range is $\mathbf{Z}^+$.

b) Since the largest decimal digit of a strictly positive integer cannot be 0, we have domain $\mathbf{Z}^+$ and range $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

c) The domain is the set of all bit strings. The number of 1's minus number of 0's can be any positive or negative integer or 0, so the range is $\mathbf{Z}$.

d) The domain is given as $\mathbf{Z}^+$. Clearly the range is $\mathbf{Z}^+$ as well.

e) The domain is the set of bit strings. The range is the set of strings of 1's, i.e., $\{\lambda, 1, 11, 111, \ldots\}$, where λ is the empty string (containing no symbols).

8. We simply round up or down in each case.
a) 1 **b)** 2 **c)** -1 **d)** 0 **e)** 3 **f)** -2 **g)** $\lfloor \frac{1}{2} + 1 \rfloor = \lfloor \frac{3}{2} \rfloor = 1$
h) $\lceil 0 + 1 + \frac{1}{2} \rceil = \lceil \frac{3}{2} \rceil = 2$

10. **a)** This is one-to-one. **b)** This is not one-to-one, since b is the image of both a and b.
c) This is not one-to-one, since d is the image of both a and d.

12. **a)** This is one-to-one, since if $n_1 - 1 = n_2 - 1$, then $n_1 = n_2$.

b) This is not one-to-one, since, for example, $f(3) = f(-3) = 10$.

c) This is one-to-one, since if $n_1^3 = n_2^3$, then $n_1 = n_2$ (take the cube root of each side).

d) This is not one-to-one, since, for example, $f(3) = f(4) = 2$.

14. **a)** This is clearly onto, since $f(0, -n) = n$ for every integer n.

b) This is not onto, since, for example, 2 is not in the range. To see this, if $m^2 - n^2 = (m-n)(m+n) = 2$, then m and n must have same parity (both even or both odd). In either case, both $m - n$ and $m + n$ are then even, so this expression is divisible by 4 and hence cannot equal 2.

c) This is clearly onto, since $f(0, n-1) = n$ for every integer n.

d) This is onto. To achieve negative values we set $m = 0$, and to achieve nonnegative values we set $n = 0$.

e) This is not onto, for the same reason as in part (b). In fact, the range here is clearly a subset of the range in that part.

16. **a)** This would normally be one-to-one, unless somehow two students in the class had a strange mobile phone service in which they shared the same phone number.

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b) This is surely one-to-one; otherwise the student identification number would not "identify" students very well!

c) This is almost surely not one-to-one; unless the class is very small, it is very likely that two students will receive the same grade.

d) This function will be one-to-one as long as no two students in the class hale from the same town (which is rather unlikely, so the function is probably not one-to-one).

18. Student answers may vary, depending on the choice of codomain.

a) A codomain could be all ten-digit positive integers; the function is not onto because there are many possible phone numbers assigned to people not in the class.

b) Under some student record systems, the student number consists of eight digits, so the codomain could be all natural numbers less than 100,000,000. The class does not have 100,000,000 students in it, so this function is not onto.

c) A codomain might be $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{F}\}$ (the answer depends on the grading system used at that school). If there were people at all five performance levels in this class, then the function would be onto. If not (for example, if no one failed the course), then it would not be onto.

d) The codomain could be the set of all cities and towns in the world. The function is clearly not onto. Alternatively, the codomain could be just the set of cities and towns from which the students in that class hale, in which case the function would be onto.

20. a) $f(n) = n + 17$ **b)** $f(n) = \lceil n/2 \rceil$

c) We let $f(n) = n - 1$ for even values of n, and $f(n) = n + 1$ for odd values of n. Thus we have $f(1) = 2$, $f(2) = 1$, $f(3) = 4$, $f(4) = 3$, and so on. Note that this is just one function, even though its definition used two formulae, depending on the the parity of n.

d) $f(n) = 17$

22. If we can find an inverse, the function is a bijection. Otherwise we must explain why the function is not on-to-one or not onto.

a) This is a bijection since the inverse function is $f^{-1}(x) = (4 - x)/3$.

b) This is not one-to-one since $f(17) = f(-17)$, for instance. It is also not onto, since the range is the interval $(-\infty, 7]$. For example, 42548 is not in the range.

c) This function is a bijection, but not from $\mathbf{R}$ to $\mathbf{R}$. To see that the domain and range are not $\mathbf{R}$, note that $x = -2$ is not in the domain, and $x = 1$ is not in the range. On the other hand, f is a bijection from $\mathbf{R} - \{-2\}$ to $\mathbf{R} - \{1\}$, since its inverse is $f^{-1}(x) = (1 - 2x)/(x - 1)$.

d) It is clear that this continuous function is increasing throughout its entire domain ($\mathbf{R}$) and it takes on both arbitrarily large values and arbitrarily small (large negative) ones. So it is a bijection. Its inverse is clearly $f^{-1}(x) = \sqrt[5]{x - 1}$.

24. The key here is that larger denominators make smaller fractions, and smaller denominators make larger fractions. We have two things to prove, since this is an "if and only if" statement. First, suppose that f is strictly increasing. This means that $f(x) < f(y)$ whenever $x < y$. To show that g is strictly decreasing, suppose that $x < y$. Then $g(x) = 1/f(x) > 1/f(y) = g(y)$. Conversely, suppose that g is strictly decreasing. This means that $g(x) > g(y)$ whenever $x < y$. To show that f is strictly increasing, suppose that $x < y$. Then $f(x) = 1/g(x) < 1/g(y) = f(y)$.

26. a) Let $f : \mathbf{R} \to \mathbf{R}$ be the given function. We are told that $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. We need to show that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. This follows immediately from the given conditions, because without loss of generality, we may assume that $x_1 < x_2$.

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b) We need to make the function increasing, but not *strictly* increasing, so, for example, we could take the trivial function $f(x) = 17$. If we want the range to be all of $\mathbf{R}$, we could define f in parts this way: $f(x) = x$ for $x < 0$; $f(x) = 0$ for $0 \le x \le 1$; and $f(x) = x - 1$ for $x > 1$.

28. For the function to be invertible, it must be a one-to-one correspondence. This means that it has to be one-to-one, which it is, and onto, which it is not, because, its range is the set of *positive* real numbers, rather than the set of *all* real numbers. When we restrict the codomain to be the set of positive real numbers, we get an invertible function. In fact, there is a well-known name for the inverse function in this case—the natural logarithm function ($g(x) = \ln x$).

30. In all parts, we simply need to compute the values $f(-1)$, $f(0)$, $f(2)$, $f(4)$, and $f(7)$ and collect the values into a set.
a) $\{1\}$ (all five values are the same) **b)** $\{-1, 1, 5, 8, 15\}$ **c)** $\{0, 1, 2\}$ **d)** $\{0, 1, 5, 16\}$

32. **a)** the set of even integers **b)** the set of positive even integers **c)** the set of real numbers

34. As in the statement, we assume for all parts that A, B, and C are sets and that f is a function from B to C and g is a function from A to B.

a) Let $z \in C$. Because $f \circ g$ is onto, there is an element $x \in A$ such that $f(g(x)) = z$. Then $g(x)$ is an element of B that maps to z under the function f, which shows that f is onto.

b) Let x_1 and x_2 be elements of A such that $g(x_1) = g(x_2)$. Then $f(g(x_1)) = f(g(x_2))$. Because $f \circ g$ is one-to-one, this implies that $x_1 = x_2$ and shows that g is one-to-one.

c) Suppose g is onto, and let y_1 and y_2 be elements of B such that $f(y_1) = f(y_2)$. Let x_1 and x_2 be elements of A such that $g(x_1) = y_1$ and $g(x_2) = y_2$. Then $(f \circ g)(x_1) = f(g(x_1)) = f(y_1) = f(y_2) = f(g(x_2)) = (f \circ g)(x_2)$, and because $f \circ g$ is one-to-one, this means that $x_1 = x_2$ and therefore $y_1 = y_2$, which shows that f is one-to-one. Conversely, suppose that f is one-to-one. To show that g is onto, let $y \in B$. Because $f \circ g$ is onto, there is an element $x \in A$ such that $f(g(x)) = f(y)$. Because f is one-to-one, this means that $y = g(x)$, which shows that g is onto.

36. To clarify the setting, suppose that $g : A \to B$ and $f : B \to C$, so that $f \circ g \colon A \to C$. We will prove that if $f \circ g$ is one-to-one, then g is also one-to-one, so not only is the answer to the question "yes," but part of the hypothesis is not even needed. Suppose that g were not one-to-one. By definition this means that there are distinct elements a_1 and a_2 in A such that $g(a_1) = g(a_2)$. Then certainly $f(g(a_1)) = f(g(a_2))$, which is the same statement as $(f \circ g)(a_1) = (f \circ g)(a_2)$. By definition this means that $f \circ g$ is not one-to-one, and our proof is complete.

38. We have $(f \circ g)(x) = f(g(x)) = f(x + 2) = (x + 2)^2 + 1 = x^2 + 4x + 5$, whereas $(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = x^2 + 1 + 2 = x^2 + 3$. Note that they are not equal.

40. Forming the compositions we have $(f \circ g)(x) = acx + ad + b$ and $(g \circ f)(x) = cax + cb + d$. These are equal if and only if $ad + b = cb + d$. In other words, equality holds for all 4-tuples (a, b, c, d) for which $ad + b = cb + d$.

42. **a)** This really has two parts. First suppose that b is in $f(S \cup T)$. Thus $b = f(a)$ for some $a \in S \cup T$. Either $a \in S$, in which case $b \in f(S)$, or $a \in T$, in which case $b \in f(T)$. Thus in either case $b \in f(S) \cup f(T)$. This shows that $f(S \cup T) \subseteq f(S) \cup f(T)$. Conversely, suppose $b \in f(S) \cup f(T)$. Then either $b \in f(S)$ or $b \in f(T)$. This means either that $b = f(a)$ for some $a \in S$ or that $b = f(a)$ for some $a \in T$. In either case, $b = f(a)$ for some $a \in S \cup T$, so $b \in f(S \cup T)$. This shows that $f(S) \cup f(T) \subseteq f(S \cup T)$, and our proof is complete.

b) Suppose $b \in f(S \cap T)$. Then $b = f(a)$ for some $a \in S \cap T$. This implies that $a \in S$ and $a \in T$, so we have $b \in f(S)$ and $b \in f(T)$. Therefore $b \in f(S) \cap f(T)$, as desired.

44. **a)** The answer is the set of all solutions to $x^2 = 1$, namely $\{1, -1\}$.

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b) In order for x^2 to be strictly between 0 and 1, we need x to be either strictly between 0 and 1 or strictly between -1 and 0. Therefore the answer is $\{\, x \mid -1 < x < 0 \vee 0 < x < 1 \,\}$.

c) In order for x^2 to be greater than 4, we need either $x > 2$ or $x < -2$. Therefore the answer is $\{\, x \mid x > 2 \vee x < -2 \,\}$.

46. a) We need to prove two things. First suppose $x \in f^{-1}(S \cup T)$. This means that $f(x) \in S \cup T$. Therefore either $f(x) \in S$ or $f(x) \in T$. In the first case $x \in f^{-1}(S)$, and in the second case $x \in f^{-1}(T)$. In either case, then, $x \in f^{-1}(S) \cup f^{-1}(T)$. Thus we have shown that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$. Conversely, suppose that $x \in f^{-1}(S) \cup f^{-1}(T)$. Then either $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$, so either $f(x) \in S$ or $f(x) \in T$. Thus we know that $f(x) \in S \cup T$, so by definition $x \in f^{-1}(S \cup T)$. This shows that $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$, as desired.

b) This is similar to part (a). We have $x \in f^{-1}(S \cap T)$ if and only if $f(x) \in S \cap T$, if and only if $f(x) \in S$ and $f(x) \in T$, if and only if $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$, if and only if $x \in f^{-1}(S) \cap f^{-1}(T)$.

48. There are three cases. Define the "fractional part" of x to be $f(x) = x - \lfloor x \rfloor$. Clearly $f(x)$ is always between 0 and 1 (inclusive at 0, exclusive at 1), and $x = \lfloor x \rfloor + f(x)$. If $f(x)$ is less than $\frac{1}{2}$, then $x + \frac{1}{2}$ will have a value slightly less than $\lfloor x \rfloor + 1$, so when we round down, we get $\lfloor x \rfloor$. In other words, in this case $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor$, and indeed that is the integer closest to x. If $f(x)$ is greater than $\frac{1}{2}$, then $x + \frac{1}{2}$ will have a value slightly greater than $\lfloor x \rfloor + 1$, so when we round down, we get $\lfloor x \rfloor + 1$. In other words, in this case $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor + 1$, and indeed that is the integer closest to x in this case. Finally, if the fractional part is exactly $\frac{1}{2}$, then x is midway between two integers, and $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor + 1$, which is the larger of these two integers.

50. If x is not an integer, then $\lceil x \rceil$ is the integer just larger than x, and $\lfloor x \rfloor$ is the integer just smaller than x. Clearly they differ by 1. If x is an integer, then $\lceil x \rceil - \lfloor x \rfloor = x - x = 0$.

52. Write $x = n - \epsilon$, where n is an integer and $0 \le \epsilon < 1$; thus $\lceil x \rceil = n$. Then $\lceil x + m \rceil = \lceil n - \epsilon + m \rceil = n + m = \lceil x \rceil + m$. Alternatively, we could proceed along the lines of the proof of identity (4a) of Table 1, shown in the text.

54. a) The "if" direction is trivial, since $x \le \lceil x \rceil$. For the other direction, suppose that $x \le n$. Since n is an integer no smaller than x, and $\lceil x \rceil$ is by definition the smallest such integer, clearly $\lceil x \rceil \le n$.

b) The "if" direction is trivial, since $\lfloor x \rfloor \le x$. For the other direction, suppose that $n \le x$. Since n is an integer not exceeding x, and $\lfloor x \rfloor$ is by definition the largest such integer, clearly $n \le \lfloor x \rfloor$.

56. To prove the first equality, write $x = n - \epsilon$, where n is an integer and $0 \le \epsilon < 1$; thus $\lceil x \rceil = n$. Therefore, $\lfloor -x \rfloor = \lfloor -n + \epsilon \rfloor = -n = -\lceil x \rceil$. The second equality is proved in the same manner, writing $x = n + \epsilon$, where n is an integer and $0 \le \epsilon < 1$. This time $\lfloor x \rfloor = n$, and $\lceil -x \rceil = \lceil -n - \epsilon \rceil = -n = -\lfloor x \rfloor$.

58. In some sense this question is its own answer—the number of integers between a and b, inclusive, is the number of integers between a and b, inclusive. Presumably we seek an expression involving a, b, and the floor and/or ceiling function to answer this question. If we round a up and round b down to integers, then we will be looking at the smallest and largest integers just inside the range of integers we want to count, respectively. These values are of course $\lceil a \rceil$ and $\lfloor b \rfloor$, respectively. Then the answer is $\lfloor b \rfloor - \lceil a \rceil + 1$ (just think of counting all the integers between these two values, including both ends—if a row of fenceposts one foot apart extends for k feet, then there are $k + 1$ fenceposts). Note that this even works when, for example, $a = 0.3$ and $b = 0.7$.

60. Since a byte is eight bits, all we are asking for in each case is $\lceil n/8 \rceil$, where n is the number of bits.
a) $\lceil 4/8 \rceil = 1$ **b)** $\lceil 10/8 \rceil = 2$ **c)** $\lceil 500/8 \rceil = 63$ **d)** $\lceil 3000/8 \rceil = 375$

62. From Example 30 we know that one ATM cell is 53 bytes, or $53 \cdot 8 = 424$ bits long. Thus in each case we need to divide the number of bits transmitted in 10 seconds by 424 and round down.

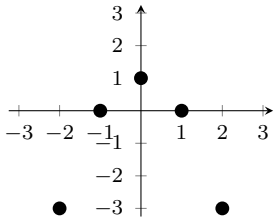
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a) In 10 seconds, this link can transmit $128{,}000 \cdot 10 = 1{,}280{,}000$ bits. So the answer is $\lfloor 1{,}280{,}000/424 \rfloor = 3018$.

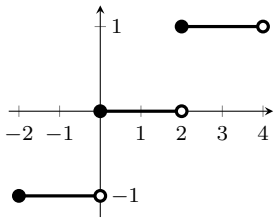
b) In 10 seconds, this link can transmit $300{,}000 \cdot 10 = 3{,}000{,}000$ bits. So the answer is $\lfloor 3{,}000{,}000/424 \rfloor = 7075$.

c) In 10 seconds, this link can transmit $1{,}000{,}000 \cdot 10 = 10{,}000{,}000$ bits. So the answer is $\lfloor 10{,}000{,}000/424 \rfloor = 23{,}584$.

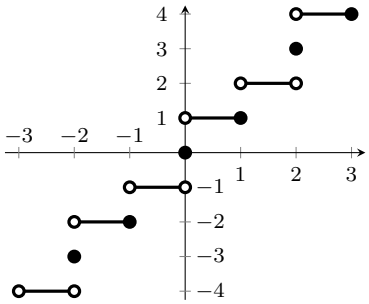
64. The graph consists of the points $(n, 1 - n^2)$ for all $n \in \mathbf{Z}$. The picture shows part of the graph on the usual coordinate axes.



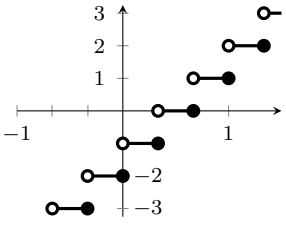
66. The graph is similar to the graph of $f(x) = \lfloor x \rfloor$; the only difference is a change in the scale of the x-axis.



68. The function values for this step function change only at integer values of x, and different things happen for odd x and for even x because of the $x/2$ term. Whatever jump pattern is established on the closed interval $[0, 2]$ must repeat indefinitely in both directions. A thoughtful analysis then yields the following graph.

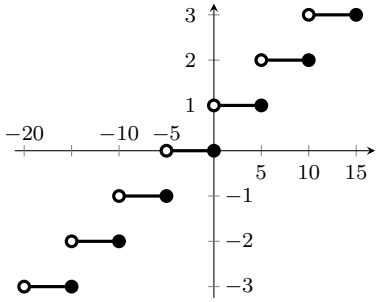


70. **a)** We can rewrite this as $f(x) = \lceil 3(x - \frac{2}{3}) \rceil$. The graph will therefore look look exactly like the graph of the function $f(x) = \lceil 3x \rceil$, except that the picture will be shifted to the right by $\frac{2}{3}$ unit, since x has been replaced by $x - \frac{2}{3}$. The graph of $f(x) = \lceil 3x \rceil$ is just like the graph shown in Figure 10b, except that the x-axis needs to be rescaled by a factor of 3 (the first jump on the positive x-axis occurs at $x = \frac{1}{3}$ here). Putting this all together yields the following picture. (Alternatively, we can think of this as the graph of $f(x) = \lceil 3x \rceil$ shifted down 2 units, since $\lceil 3x - 2 \rceil = \lceil 3x \rceil - 2$.)

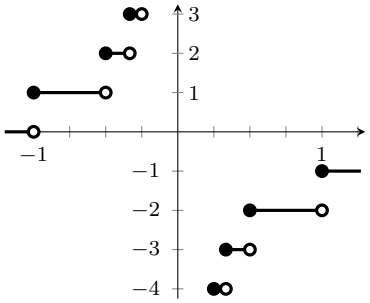


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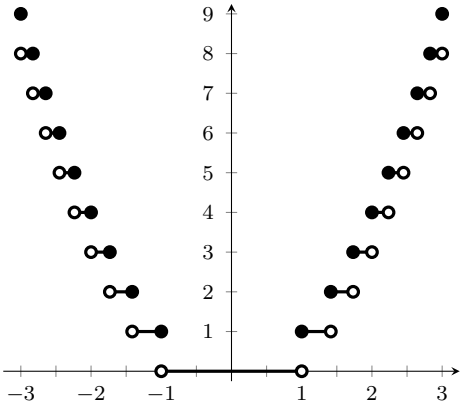
b) The graph will look exactly like the graph shown in Figure 10b, except that the x-axis needs to be rescaled by a factor of 5 (the first jump on the positive x-axis occurs at $x = 5$ here).



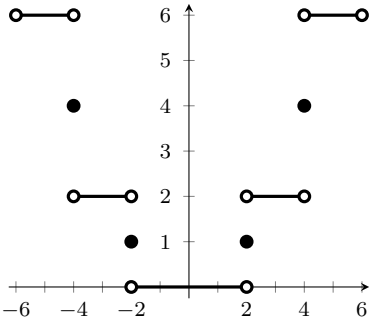
c) Since $\lfloor -1/x \rfloor = -\lceil 1/x \rceil$ (see Exercise 56), the picture is just the picture for Exercise 69d flipped upside down.



d) The basic shape is the parabola, $y = x^2$. However, because of the greatest integer function, the curve is broken into steps, with jumps at $x = \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \ldots$. Note the symmetry around the y-axis.

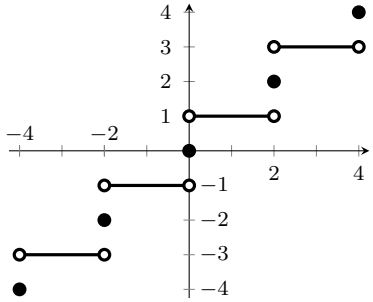


e) The basic shape is the parabola, $y = x^2/4$. However, because of the step functions, the curve is broken into steps. For x an even integer, $f(x) = x^4/4$, since the terms inside the floor and ceiling function symbols are integers. Note how these are isolated point, as in Exercise 69f.

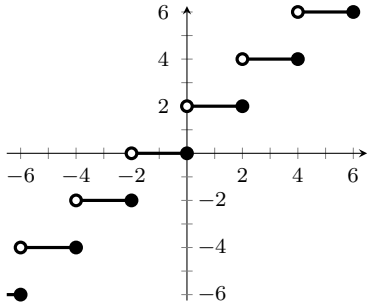


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f) When x is an even integer, this is just x. When x is between two even integers, however, this has the value of the odd integer between them. The graph is therefore as shown here.



g) Despite the complicated-looking formula, this is not too hard. Note that the expression inside the outer floor function symbols is always going to be an integer plus $\frac{1}{2}$; therefore we can tell exactly what its rounded-down value will be, namely $2\lceil x/2 \rceil$. This is just the graph in Figure 10b, rescaled on both axes.



72. This follows immediately from the definition. We want to show that $\big((f \circ g) \circ (g^{-1} \circ f^{-1})\big)(z) = z$ for all $z \in Z$ and that $\big((g^{-1} \circ f^{-1}) \circ (f \circ g)\big)(x) = x$ for all $x \in X$. For the first we have

$$\begin{aligned}\big((f \circ g) \circ (g^{-1} \circ f^{-1})\big)(z) &= (f \circ g)((g^{-1} \circ f^{-1})(z)) \\ &= (f \circ g)(g^{-1}(f^{-1}(z))) \\ &= f(g(g^{-1}(f^{-1}(z)))) \\ &= f(f^{-1}(z)) = z\,.\end{aligned}$$

The second equality is similar.

74. If f is one-to-one, then every element of A gets sent to a different element of B. If in addition to the range of A there were another element in B, then $|B|$ would be at least one greater than $|A|$. This cannot happen, so we conclude that f is onto. Conversely, suppose that f is onto, so that every element of B is the image of some element of A. In particular, there is an element of A for each element of B. If two or more elements of A were sent to the same element of B, then $|A|$ would be at least one greater than the $|B|$. This cannot happen, so we conclude that f is one-to-one.

76. **a)** This is true. Since $\lceil x \rceil$ is already an integer, $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$.

b) A little experimentation shows that this is not always true. To disprove it we need only produce a counterexample, such as $x = y = \frac{3}{4}$. In this case the left-hand side is $\lfloor 3/2 \rfloor = 1$, while the right-hand side is $0 + 0 = 0$.

c) A little trial and error fails to produce a counterexample, so maybe this is true. We look for a proof. Since we are dividing by 4, let us write $x = 4n + k$, where $0 \le k < 4$. In other words, write x in terms of how much it exceeds the largest multiple of 4 not exceeding it. There are three cases. If $k = 0$, then x is already a multiple of 4, so both sides equal n. If $0 < k \le 2$, then $\lceil x/2 \rceil = 2n + 1$, so the left-hand side is

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$\lceil n+\frac{1}{2}\rceil = n+1$. Of course the right-hand side is $n+1$ as well, so again the two sides agree. Finally, suppose that $2 < k < 4$. Then $\lceil x/2\rceil = 2n+2$, and the left-hand side is $\lceil n+1\rceil = n+1$; of course the right-hand side is still $n+1$, as well. Since we proved that the two sides are equal in all cases, the proof is complete.

d) For $x = 8.5$, the left-hand side is 3, whereas the right-hand side is 2.

e) This is true. Write $x = n+\epsilon$ and $y = m+\delta$, where n and m are integers and ϵ and δ are nonnegative real numbers less than 1. The left-hand side is $n+m+(n+m)$ or $n+m+(n+m+1)$, the latter occurring if and only if $\epsilon+\delta \geq 1$. The right-hand side is the sum of two quantities. The first is either $2n$ (if $\epsilon < \frac{1}{2}$) or $2n+1$ (if $\epsilon \geq \frac{1}{2}$). The second is either $2m$ (if $\delta < \frac{1}{2}$) or $2m+1$ (if $\delta \geq \frac{1}{2}$). The only way, then, for the left-hand side to exceed the right-hand side is to have the left-hand side be $2n+2m+1$ and the right-hand side be $2n+2m$. This can occur only if $\epsilon+\delta \geq 1$ while $\epsilon < \frac{1}{2}$ and $\delta < \frac{1}{2}$. But that is an impossibility, since the sum of two numbers less than $\frac{1}{2}$ cannot be as large as 1. Therefore the right-hand side is always at least as large as the left-hand side.

78. A straightforward way to do this problem is to consider the three cases determined by where in the interval between two consecutive integers the real number x lies. Certainly every real number x lies in an interval $[n, n+1)$ for some integer n; indeed, $n = \lfloor x\rfloor$. (Recall that $[s,t)$ is the notation for the set of real numbers greater than or equal to s and less than t.) If $x \in [n, n+\frac{1}{3})$, then $3x$ lies in the interval $[3n, 3n+1)$, so $\lfloor 3x\rfloor = 3n$. Moreover in this case $x+\frac{1}{3}$ is still less than $n+1$, and $x+\frac{2}{3}$ is still less than $n+1$, so $\lfloor x\rfloor + \lfloor x+\frac{1}{3}\rfloor + \lfloor x+\frac{2}{3}\rfloor = n+n+n = 3n$ as well. For the second case, we assume that $x \in [n+\frac{1}{3}, n+\frac{2}{3})$. This time $3x \in [3n+1, 3n+2)$, so $\lfloor 3x\rfloor = 3n+1$. Moreover in this case $x+\frac{1}{3}$ is in $[n+\frac{2}{3}, n+1)$, and $x+\frac{2}{3}$ is in $[n+1, n+\frac{4}{3})$, so $\lfloor x\rfloor + \lfloor x+\frac{1}{3}\rfloor + \lfloor x+\frac{2}{3}\rfloor = n+n+(n+1) = 3n+1$ as well. The third case, $x \in [n+\frac{2}{3}, n+1)$, is similar, with both sides equaling $3n+2$.

80. a) We merely have to remark that f^* is well-defined by the rule given here. For each $a \in A$, either a is in the domain of definition of f or it is not. If it is, then $f^*(a)$ is the well-defined element $f(a) \in B$, and otherwise $f^*(a) = u$. In either case $f^*(a)$ is a well-defined element of $B \cup \{u\}$.

b) We simply need to set $f^*(a) = u$ for each a not in the domain of definition of f. In part (a), then, $f^*(n) = 1/n$ for $n \neq 0$, and $f^*(0) = u$. In part (b) we have a total function already, so $f^*(n) = \lceil n/2\rceil$ for all $n \in \mathbf{Z}$. In part (c) $f^*(m,n) = m/n$ if $n \neq 0$, and $f^*(m,0) = u$ for all $m \in \mathbf{Z}$. In part (d) we have a total function already, so $f^*(m,n) = mn$ for all values of m and n. In part (e) the rule only applies if $m > n$, so $f^*(m,n) = m-n$ if $m > n$, and $f^*(m,n) = u$ if $m \leq n$.

82. For the "if" direction, we simply need to note that if S is a finite set, with cardinality m, then every proper subset of S has cardinality strictly smaller than m, so there is no possible one-to-one correspondence between the elements of S and the elements of the proper subset. (This is essentially the pigeonhole principle, to be discussed in Section 6.2.)

The "only if" direction is much deeper. Let S be the given infinite set. Clearly S is not empty, because by definition, the empty set has cardinality 0, a nonnegative integer. Let a_0 be one element of S, and let $A = S - \{a_0\}$. Clearly A is also infinite (because if it were finite, then we would have $|S| = |A|+1$, making S finite). We will now construct a one-to-one correspondence between S and A; think of this as a one-to-one and onto function f from S to A. (This construction is an infinite process; technically we are using something called the Axiom of Choice.) In order to define $f(a_0)$, we choose an arbitrary element a_1 in A (which is possible because A is infinite) and set $f(a_0) = a_1$. Next we define f at a_1. To do so, we choose an arbitrary element a_2 in $A - \{a_1\}$ (which is possible because $A - \{a_1\}$ is necessarily infinite) and set $f(a_1) = a_2$. Next we define f at a_2. To do so, we choose an arbitrary element a_3 in $A - \{a_1, a_2\}$ (which is possible because $A - \{a_1, a_2\}$ is necessarily infinite) and set $f(a_2) = a_3$. We continue this process forever. Finally, we let f be the identity function on $S - \{a_0, a_1, a_2, \ldots\}$. The function thus defined has $f(a_i) = a_{i+1}$ for all natural numbers i and $f(x) = x$ for all $x \in S - \{a_0, a_1, a_2, \ldots\}$. Our construction forced f to be one-to-one and onto.

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SECTION 2.4 Sequences and Summations

2. In each case we just plug $n = 8$ into the formula.
a) $2^{8-1} = 128$ **b)** 7 **c)** $1 + (-1)^8 = 2$ **d)** $-(-2)^8 = -256$

4. **a)** $a_0 = (-2)^0 = 1,\ a_1 = (-2)^1 = -2,\ a_2 = (-2)^2 = 4,\ a_3 = (-2)^3 = -8$

b) $a_0 = a_1 = a_2 = a_3 = 3$

c) $a_0 = 7 + 4^0 = 8,\ a_1 = 7 + 4^1 = 11,\ a_2 = 7 + 4^2 = 23,\ a_3 = 7 + 4^3 = 71$

d) $a_0 = 2^0 + (-2)^0 = 2,\ a_1 = 2^1 + (-2)^1 = 0,\ a_2 = 2^2 + (-2)^2 = 8,\ a_3 = 2^3 + (-2)^3 = 0$

6. These are easy to compute by hand, calculator, or computer.
a) 10, 7, 4, 1, -2, -5, -8, -11, -14, -17

b) We can use the formula in Table 2, or we can just keep adding to the previous term ($1+2=3$, $3+3=6$, $6+4=10$, and so on): 1, 3, 6, 10, 15, 21, 28, 36, 45, 55. These are called the triangular numbers.

c) 1, 5, 19, 65, 211, 665, 2059, 6305, 19171, 58025

d) 1, 1, 1, 2, 2, 2, 2, 2, 3, 3 (there will be $2k+1$ copies of k) **e)** 1, 5, 6, 11, 17, 28, 45, 73, 118, 191

f) The largest number whose binary expansion has n bits is $(11\ldots1)_2$, which is $2^n - 1$. So the sequence is 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023.

g) 1, 2, 2, 4, 8, 11, 33, 37, 148, 153 **h)** 1, 2, 2, 2, 2, 3, 3, 3, 3, 3

8. One rule could be that each term is 2 greater than the previous term; the sequence would be 3, 5, 7, 9, 11, 13, ... Another rule could be that the n^{th} term is the n^{th} odd prime; the sequence would be 3, 5, 7, 11, 13, 17, ... Actually, we could choose any number we want for the fourth term (say 12) and find a third degree polynomial whose value at n would be the n^{th} term; in this case we need to solve for A, B, C, and D in the equations $y = Ax^3 + Bx^2 + Cx + D$ where $(1,3)$, $(2,5)$, $(3,7)$, $(4,12)$ have been plugged in for x and y. Doing so yields $(x^3 - 6x^2 + 15x - 4)/2$. With this formula, the sequence is 3, 5, 7, 12, 23, 43, 75, 122, 187, 273. Obviously many other answers are possible.

10. In each case we simply plug $n = 0, 1, 2, 3, 4, 5$, using the initial conditions for the first few and then the recurrence relation.

a) $a_0 = -1,\ a_1 = -2a_0 = 2,\ a_2 = -2a_1 = -4,\ a_3 = -2a_2 = 8,\ a_4 = -2a_3 = -16,\ a_5 = -2a_4 = 32$

b) $a_0 = 2,\ a_1 = -1,\ a_2 = a_1 - a_0 = -3,\ a_3 = a_2 - a_1 = -2,\ a_4 = a_3 - a_2 = 1,\ a_5 = a_4 - a_3 = 3$

c) $a_0 = 1,\ a_1 = 3a_0^2 = 3,\ a_2 = 3a_1^2 = 27 = 3^3,\ a_3 = 3a_2^2 = 2187 = 3^7,\ a_4 = 3a_3^2 = 14348907 = 3^{15},\ a_5 = 3a_4^2 = 617673396283947 = 3^{31}$

d) $a_0 = -1,\ a_1 = 0,\ a_2 = 2a_1 + a_0^2 = 1,\ a_3 = 3a_2 + a_1^2 = 3,\ a_4 = 4a_3 + a_2^2 = 13,\ a_5 = 5a_4 + a_3^2 = 74$

e) $a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = a_2 - a_1 + a_0 = 2\ ,\ a_4 = a_3 - a_2 + a_1 = 1\ ,\ a_5 = a_4 - a_3 + a_2 = 1$

12. **a)** $-3a_{n-1} + 4a_{n-2} = -3 \cdot 0 + 4 \cdot 0 = 0 = a_n$ **b)** $-3a_{n-1} + 4a_{n-2} = -3 \cdot 1 + 4 \cdot 1 = 1 = a_n$

c) $-3a_{n-1} + 4a_{n-2} = -3 \cdot (-4)^{n-1} + 4 \cdot (-4)^{n-2} = (-4)^{n-2}\big((-3)(-4) + 4\big) = (-4)^{n-2} \cdot 16 = (-4)^{n-2}(-4)^2 = (-4)^n = a_n$

d) $-3a_{n-1} + 4a_{n-2} = -3 \cdot \big(2(-4)^{n-1} + 3\big) + 4 \cdot \big(2(-4)^{n-2} + 3\big) = (-4)^{n-2}\big((-6)(-4) + 4 \cdot 2\big) - 9 + 12 = (-4)^{n-2} \cdot 32 + 3 = (-4)^{n-2}(-4)^2 \cdot 2 + 3 = 2 \cdot (-4)^n + 3 = a_n$

14. In each case, one possible answer is just the equation as presented (it is a recurrence relation of degree 0). We will give an alternate answer.

a) One possible answer is $a_n = a_{n-1}$.

b) Note that $a_n - a_{n-1} = 2n - (2n-2) = 2$. Therefore we have $a_n = a_{n-1} + 2$ as one possible answer.

c) Just as in part (b), we have $a_n = a_{n-1} + 2$.

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d) Probably the simplest answer is $a_n = 5a_{n-1}$.

e) Since $a_n - a_{n-1} = n^2 - (n-1)^2 = 2n-1$, we have $a_n = a_{n-1} + 2n - 1$.

f) This is similar to part (e). One answer is $a_n = a_{n-1} + 2n$.

g) Note that $a_n - a_{n-1} = n + (-1)^n - (n-1) - (-1)^{n-1} = 1 + 2(-1)^n$. Thus we have $a_n = a_{n-1} + 1 + 2(-1)^n$.

h) $a_n = na_{n-1}$

16. In the iterative approach, we write a_n in terms of a_{n-1}, then write a_{n-1} in terms of a_{n-2} (using the recurrence relation with $n-1$ plugged in for n), and so on. When we reach the end of this procedure, we use the given initial value of a_0. This will give us an explicit formula for the answer or it will give us a finite series, which we then sum to obtain an explicit formula for the answer.

a) $a_n = -a_{n-1} = (-1)^2 a_{n-2} = \cdots = (-1)^n a_{n-n} = (-1)^n a_0 = 5 \cdot (-1)^n$

b) $a_n = 3 + a_{n-1} = 3 + 3 + a_{n-2} = 2 \cdot 3 + a_{n-2} = 3 \cdot 3 + a_{n-3} = \cdots = n \cdot 3 + a_{n-n} = n \cdot 3 + a_0 = 3n + 1$

c) $a_n = -n + a_{n-1}$
$= -n + \big(-(n-1) + a_{n-2}\big) = -\big(n + (n-1)\big) + a_{n-2}$
$= -\big(n + (n-1)\big) + \big(-(n-2) + a_{n-3}\big) = -\big(n + (n-1) + (n-2)\big) + a_{n-3}$
$\vdots$
$= -\big(n + (n-1) + (n-2) + \cdots + (n - (n-1))\big) + a_{n-n}$
$= -\big(n + (n-1) + (n-2) + \cdots + 1\big) + a_0$
$= -\dfrac{n(n+1)}{2} + 4 = \dfrac{-n^2 - n + 8}{2}$

d) $a_n = -3 + 2a_{n-1}$
$= -3 + 2(-3 + 2a_{n-2}) = -3 + 2(-3) + 4a_{n-2}$
$= -3 + 2(-3) + 4(-3 + 2a_{n-3}) = -3 + 2(-3) + 4(-3) + 8a_{n-3}$
$= -3 + 2(-3) + 4(-3) + 8(-3 + 2a_{n-4}) = -3 + 2(-3) + 4(-3) + 8(-3) + 16a_{n-4}$
$\vdots$
$= -3(1 + 2 + 4 + \cdots + 2^{n-1}) + 2^n a_{n-n} = -3(2^n - 1) + 2^n(-1) = -2^{n+2} + 3$

e) $a_n = (n+1)a_{n-1} = (n+1)na_{n-2}$
$= (n+1)n(n-1)a_{n-3} = (n+1)n(n-1)(n-2)a_{n-4}$
$\vdots$
$= (n+1)n(n-1)(n-2)(n-3)\cdots(n-(n-2))\,a_{n-n}$
$= (n+1)n(n-1)(n-2)(n-3)\cdots 2 \cdot a_0$
$= (n+1)! \cdot 2 = 2(n+1)!$

f) $a_n = 2na_{n-1}$
$= 2n\big(2(n-1)a_{n-2}\big) = 2^2\big(n(n-1)\big)a_{n-2}$
$= 2^2\big(n(n-1)\big)\big(2(n-2)a_{n-3}\big) = 2^3\big(n(n-1)(n-2)\big)a_{n-3}$
$\vdots$
$= 2^n n(n-1)(n-2)(n-3)\cdots\big(n-(n-1)\big)a_{n-n}$
$= 2^n n(n-1)(n-2)(n-3)\cdots 1 \cdot a_0$
$= 3 \cdot 2^n n!$

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g) $a_n = n - 1 - a_{n-1}$
$= n - 1 - \big((n-1-1) - a_{n-2}\big) = (n-1) - (n-2) + a_{n-2}$
$= (n-1) - (n-2) + \big((n-2-1) - a_{n-3}\big) = (n-1) - (n-2) + (n-3) - a_{n-3}$
$\vdots$
$= (n-1) - (n-2) + \cdots + (-1)^{n-1}(n-n) + (-1)^n a_{n-n}$
$= \dfrac{2n - 1 + (-1)^n}{4} + (-1)^n \cdot 7$

18. **a)** The amount after $n-1$ years is multiplied by 1.09 to give the amount after n years, since 9% of the value must be added to account for the interest. Thus we have $a_n = 1.09a_{n-1}$. The initial condition is $a_0 = 1000$.

b) Since we multiply by 1.09 for each year, the solution is $a_n = 1000(1.09)^n$.

c) $a_{100} = 1000(1.09)^{100} \approx \$5{,}529{,}041$

20. This is just like Exercise 18. We are letting a_n be the population, in billions of people, n years after 2017.

a) $a_n = 1.0112a_{n-1}$, with $a_0 = 7.6$ **b)** $a_n = 7.6 \cdot (1.0112)^n$

c) $a_{33} = 7.6 \cdot (1.0112)^{33} \approx 10.98$ billion people

22. We let a_n be the salary, in thousands of dollars, n years after the employee started at the company.

a) $a_n = 1 + 1.05a_{n-1}$, with $a_0 = 50$

b) Here $n = 8$. We can either iterate the recurrence relation 8 times, or we can use the result of part (c). The answer turns out to be approximately $a_8 = 83.4$, i.e., a salary of approximately \$83,400.

c) We use the iterative approach.

$$\begin{aligned} a_n &= 1 + 1.05a_{n-1} \\ &= 1 + 1.05(1 + 1.05a_{n-2}) \\ &= 1 + 1.05 + (1.05)^2 a_{n-2} \\ &\vdots \\ &= 1 + 1.05 + (1.05)^2 + \cdots + (1.05)^{n-1} + (1.05)^n a_0 \\ &= \frac{(1.05)^n - 1}{1.05 - 1} + 50 \cdot (1.05)^n \\ &= 70 \cdot (1.05)^n - 20 \end{aligned}$$

24. **a)** Each month our account accrues some interest that must be paid. Since the balance the previous month is $B(k-1)$, the amount of interest we owe is $(r/12)B(k-1)$. After paying this interest, the rest of the P dollar payment we make each month goes toward reducing the principle. Therefore we have $B(k) = B(k-1) - (P - (r/12)B(k-1))$. This can be simplified to $B(k) = (1 + (r/12))B(k-1) - P$. The initial condition is that $B(0) =$ the amount borrowed.

b) Solving this by iteration yields

$$B(k) = (1 + (r/12))^k (B(0) - 12P/r) + 12P/r\,.$$

Setting $B(k) = 0$ and solving this for k yields the desired value of T after some messy algebra, namely

$$T = \frac{\log(-12P/(B(0)r - 12P))}{\log(1 + (r/12))}\,.$$

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26. **a)** The first term is 3, and the n^{th} term is obtained by adding $2n-1$ to the previous term. In other words, we successively add 3, then 5, then 7, and so on. Alternatively, we see that the n^{th} term is n^2+2; we can see this by inspection if we happen to notice how close each term is to a perfect square, or we can fit a quadratic polynomial to the data. The next three terms are 123, 146, 171.

b) This is an arithmetic sequence whose first term is 7 and whose difference is 4. Thus the n^{th} term is $7+4(n-1)=4n+3$. Thus the next three terms are 47, 51, 55.

c) The n^{th} term is clearly the binary expansion of n. Thus the next three terms are 1100, 1101, 1110.

d) The sequence consists of one 1, followed by three 2's, followed by five 3's, followed by seven 5's, and so on, with the number of copies of the next value increasing by 2 each time, and the values themselves following the rule that the first two values are 1 and 2 and each subsequent value is the sum of the previous two values. Obviously other answers are possible as well. By our rule, the next three terms would be 8, 8, 8.

e) If we stare at this sequence long enough and compare it with Table 1, then we notice that the n^{th} term is 3^n-1. Thus the next three terms are 59048, 177146, 531440.

f) We notice that each term evenly divides the next, and the multipliers are successively 3, 5, 7, 9, 11, and so on. That must be the intended pattern. One notation for this is to use $n!!$ to mean $n(n-2)(n-4)\cdots$; thus the n^{th} term is $(2n-1)!!$. Thus the next three terms are 654729075, 13749310575, 316234143225.

g) The sequence consists of one 1, followed by two 0s, then three 1s, four 0s, five 1s, and so on, alternating between 0s and 1s and having one more item in each group than in the previous group. Thus six 0's will follow next, so the next three terms are 0, 0, 0.

h) It doesn't take long to notice that each term is the square of its predecessor. The next three terms get very big very fast: 18446744073709551616, $340282366920938463463374607431768211456$, and then

$$115792089237316195423570985008687907853269984665640564039457584007913129639936\,.$$

(These were computed using *Maple*.)

28. Let us ask ourselves which is the last term in the sequence whose value is k? Clearly it is $1+2+3+\cdots+k$, which equals $k(k+1)/2$. We can rephrase this by saying that $a_n \le k$ if and only if $k(k+1)/2 \ge n$. Thus, to find k as a function of n, we must find the smallest k such that $k(k+1)/2 \ge n$. This is equivalent to $k^2+k-2n \ge 0$. By the quadratic formula, this tells us that k has to be at least $(-1+\sqrt{1+8n})/2$. Therefore we have $k = \lceil(-1+\sqrt{1+8n})/2\rceil = \left\lceil -\frac{1}{2}+\sqrt{2n+\frac{1}{4}}\right\rceil$. By Exercise 49 in Section 2.3, this is the same as the integer closest to $\sqrt{2n+\frac{1}{4}}$, where we choose the smaller of the two closest integers if $\sqrt{2n+\frac{1}{4}}$ is a half integer. The desired answer is $\left\lfloor\sqrt{2n}+\frac{1}{2}\right\rfloor$, which by Exercise 48 in Section 2.3 is the integer closest to $\sqrt{2n}$ (note that $\sqrt{2n}$ can never be a half integer). To see that these are the same, note that it can never happen that $\sqrt{2n} \le m+\frac{1}{2}$ while $\sqrt{2n+\frac{1}{4}} > m+\frac{1}{2}$ for some positive integer m, since this would imply that $2n \le m^2+m+\frac{1}{4}$ and $2n > m^2+m$, an impossibility. Therefore the integer closest to $\sqrt{2n}$ and the (smaller) integer closest to $\sqrt{2n+\frac{1}{4}}$ are the same, and we are done.

30. **a)** $1+3+5+7=16$ **b)** $1^2+3^2+5^2+7^2=84$
c) $(1/1)+(1/3)+(1/5)+(1/7)=176/105$ **d)** $1+1+1+1=4$

32. **a)** The terms of this sequence alternate between 2 (if j is even) and 0 (if j is odd). Thus the sum is $2+0+2+0+2+0+2+0+2=10$.

b) We can break this into two parts and compute $\left(\sum_{j=0}^{8} 3^j\right)-\left(\sum_{j=0}^{8} 2^j\right)$. Each summation can be computed from the formula for the sum of a geometric progression. Thus the answer is

$$\frac{3^9-1}{3-1}-\frac{2^9-1}{2-1}=9841-511=9330\,.$$

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c) As in part (b) we can break this into two parts and compute $\left(\sum_{j=0}^{8} 2\cdot 3^j\right)+\left(\sum_{j=0}^{8} 3\cdot 2^j\right)$. Each summation can be computed from the formula for the sum of a geometric progression. Thus the answer is

$$\frac{2\cdot 3^9-2}{3-1}+\frac{3\cdot 2^9-3}{2-1}=19682+1533=21215\,.$$

d) This could be worked as in part (b), but it is easier to note that the sum telescopes (see Exercise 35). Each power of 2 cancels except for the -2^0 when $j=0$ and the 2^9 when $j=8$. Therefore the answer is $2^9-2^0=511$. (Alternatively, note that $2^{j+1}-2^j=2^j$.)

34. We will just write out the sums explicitly in each case.

a) $(1-1)+(1-2)+(2-1)+(2-2)+(3-1)+(3-2)=3$

b) $(0+0)+(0+2)+(0+4)+(3+0)+(3+2)+(3+4)+(6+0)+(6+2)+(6+4)+(9+0)+(9+2)+(9+4)=78$

c) $(0+1+2)+(0+1+2)+(0+1+2)=9$

d) $(0+0+0+0)+(0+1+8+27)+(0+4+32+108)=180$

36. We use the suggestion (simple algebra shows that this is indeed an identity) and note that all the terms in the summation cancel out except for the $1/k$ when $k=1$ and the $1/(k+1)$ when $k=n$:

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\frac{1}{1}-\frac{1}{n+1}=\frac{n}{n+1}$$

38. First we note that $k^3-(k-1)^3=3k^2-3k+1$. Then we sum this equation for all values of k from 1 to n. On the left, because of telescoping, we have just n^3; on the right we have

$$3\sum_{k=1}^{n}k^2-3\sum_{k=1}^{n}k+\sum_{k=1}^{n}1=3\sum_{k=1}^{n}k^2-\frac{3n(n+1)}{2}+n\,.$$

Equating the two sides and solving for $\sum_{k=1}^{n}k^2$, we obtain the desired formula.

$$\begin{aligned}\sum_{k=1}^{n}k^2&=\frac{1}{3}\left(n^3+\frac{3n(n+1)}{2}-n\right)\\&=\frac{n}{3}\left(\frac{2n^2+3n+3-2}{2}\right)\\&=\frac{n}{3}\left(\frac{2n^2+3n+1}{2}\right)=\frac{n(n+1)(2n+1)}{6}\end{aligned}$$

40. This exercise is like Example 23. From Table 2 we know that $\sum_{k=1}^{200}k^3=200^2\cdot 201^2/4=404{,}010{,}000$, and $\sum_{k=1}^{98}k^3=98^2\cdot 99^2/4=23{,}532{,}201$. Therefore the desired sum is $404{,}010{,}000-23{,}532{,}201=380{,}477{,}799$.

42. In order to apply the formulas from Table 2, we first simplify the summand and rewrite the summation as several summations for which we have formulae.

$$\begin{aligned}\sum_{k=10}^{20}(k-1)(2k^2+1)&=\sum_{k=10}^{20}2k^3-2k^2+k-1\\&=2\left(\sum_{k=10}^{20}k^3\right)-2\left(\sum_{k=10}^{20}k^2\right)+\left(\sum_{k=10}^{20}k\right)-\left(\sum_{k=10}^{20}1\right)\end{aligned}$$

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We compute each summation in turn, following Example 23.

$$\sum_{k=10}^{20} k^3 = \sum_{k=1}^{20} k^3 - \sum_{k=1}^{9} k^3 = \frac{20^2(20+1)^2}{4} - \frac{9^2(9+1)^2}{4} = 44100 - 2025 = 42075$$

$$\sum_{k=10}^{20} k^2 = \sum_{k=1}^{20} k^2 - \sum_{k=1}^{9} k^2 = \frac{20(20+1)(2\cdot 20+1)}{6} - \frac{9(9+1)(2\cdot 9+1)}{6} = 2870 - 285 = 2585$$

$$\sum_{k=10}^{20} k = \sum_{k=1}^{20} k - \sum_{k=1}^{9} k = \frac{20(20+1)}{2} - \frac{9(9+1)}{2} = 210 - 45 = 165$$

$$\sum_{k=10}^{20} 1 = 11$$

Putting them together, we have $2 \cdot 42075 - 2 \cdot 2585 + 165 - 11 = 79134$.

44. If we write down the first few terms of this sum we notice a pattern. It starts $(1+1+1+1+1+1+1+1)+(2+2+2+2+2+2+2+2+2+2+2+2+2+2+2+2+2+2+2+2+2)+(3+3+3+3+\cdots+3)+\cdots$. There are seven 1s, then 19 2s, then 37 3s, and so on; in general, the number of i's is $(i+1)^3 - i^3 = 3i^2+3i+1$. So we need to sum $i(3i^2+3i+1)$ for an appropriate range of values for i. We must find this range. It gets a little messy at the end if m is such that the sequence stops before a complete range of the last value is present. Let $n = \lfloor \sqrt[3]{m} \rfloor - 1$. Then there are $n+1$ blocks, and $(n+1)^3 - 1$ is where the next-to-last block ends. The sum of those complete blocks is $\sum_{i=1}^{n} i(3i^2+3i+1) = \sum_{i=1}^{n} 3i^3+3i^2+i = n(3n+4)(n+1)^2/4$ (using Table 2 and algebra). The remaining terms in our summation all have the value $n+1$ and the number of them present is $m - ((n+1)^3 - 1)$. Our final answer is therefore $n(3n+4)(n+1)^2/4 + (n+1)(m-(n+1)^3+1)$, where, once again, $n = \lfloor \sqrt[3]{m} \rfloor - 1$.

46. $n! = \prod_{i=1}^{n} i$

48. $(0!)(1!)(2!)(3!)(4!) = 1 \cdot 1 \cdot 2 \cdot 6 \cdot 24 = 288$

SECTION 2.5 Cardinality of Sets

2. **a)** This set is countably infinite. The integers in the set are 11, 12, 13, 14, and so on. We can list these numbers in that order, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 11$, $2 \leftrightarrow 12$, $3 \leftrightarrow 13$, and so on; in general $n \leftrightarrow (n+10)$.

b) This set is countably infinite. The integers in the set are -1, -3, -5, -7, and so on. We can list these numbers in that order, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow -1$, $2 \leftrightarrow -3$, $3 \leftrightarrow -5$, and so on; in general $n \leftrightarrow -(2n-1)$.

c) This set is $\{-999{,}999, -999{,}998, \ldots, -1, 0, 1, \ldots, 999{,}999\}$. It is finite, with cardinality 1,999,999.

d) This set is uncountable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable in Example 5.

e) This set is countable. We can list its elements in the order $(2,1), (3,1), (2,2), (3,2), (2,3), (3,3), \ldots$, giving us the one-to-one correspondence $1 \leftrightarrow (2,1), 2 \leftrightarrow (3,1), 3 \leftrightarrow (2,2), 4 \leftrightarrow (3,2), 5 \leftrightarrow (2,3), 6 \leftrightarrow (3,3), \ldots$.

f) This set is countable. The integers in the set are 0, ± 10, ± 20, ± 30, and so on. We can list these numbers in the order 0, 10, -10, 20, -20, 30, ..., thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 0$, $2 \leftrightarrow 10$, $3 \leftrightarrow -10$, $4 \leftrightarrow 20$, $5 \leftrightarrow -20$, $6 \leftrightarrow 30$, and so on.

4. **a)** This set is countable. The integers in the set are ± 1, ± 2, ± 4, ± 5, ± 7, and so on. We can list these numbers in the order 1, -1, 2, -2, 4, -4, 5, -5, 7, -7, ..., thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 1$, $2 \leftrightarrow -1$, $3 \leftrightarrow 2$, $4 \leftrightarrow -2$, $5 \leftrightarrow 4$, and so on.

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b) This is similar to part (a); we can simply list the elements of the set in order of increasing absolute value, listing each positive term before its corresponding negative: $5, -5, 10, -10, 15, -15, 20, -20, 25, -25, 30, -30, 40, -40, 45, -45, 50, -50, \ldots$.

c) This set is countable but a little tricky. We can arrange the numbers in a 2-dimensional table as follows:

$.\overline{1}$	.1	.11	.111	.1111	.11111	.111111	...
$1.\overline{1}$	1	1.1	1.11	1.111	1.1111	1.11111	...
$11.\overline{1}$	11	11.1	11.11	11.111	11.1111	11.11111	...
$111.\overline{1}$	111	111.1	111.11	111.111	111.1111	111.11111	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Thus we have shown that our set is the countable union of countable sets (each of the countable sets is one row of this table). Therefore by Exercise 27, the entire set is countable. For an explicit correspondence with the positive integers, we can zigzag along the positive-sloping diagonals as in Figure 3: $1 \leftrightarrow .\overline{1}$, $2 \leftrightarrow 1.\overline{1}$, $3 \leftrightarrow .1$, $4 \leftrightarrow .11$, $5 \leftrightarrow 1$, and so on.

d) This set is not countable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable in Example 5. All we need to do is choose $d_i = 1$ when $d_{ii} = 9$ and choose $d_i = 9$ when $d_{ii} = 1$ or d_{ii} is blank (if the decimal expansion is finite).

6. We want a one-to-one function from the set of positive integers to the set of odd positive integers. The simplest one to use is $f(n) = 2n - 1$. We put the guest currently in Room n into Room $(2n - 1)$. Thus the guest in Room 1 stays put, the guest in Room 2 moves to Room 3, the guest in Room 3 moves to Room 5, and so on.

8. First we can make the move explained in Exercise 6, which frees up all the even-numbered rooms. The new guests can go into those rooms (the first into Room 2, the second into Room 4, and so on).

10. In each case, let us take A to be the set of real numbers.

a) We can let B be the set of real numbers as well; then $A - B = \emptyset$, which is finite.

b) We can let B be the set of real numbers that are not positive integers; in symbols, $B = A - \mathbf{Z}^+$. Then $A - B = \mathbf{Z}^+$, which is countably infinite.

c) We can let B be the set of positive real numbers. Then $A - B$ is the set of negative real numbers and 0, which is certainly uncountable.

12. The definition of $|A| \leq |B|$ is that there is a one-to-one function from A to B. In this case the desired function is just $f(x) = x$ for each $x \in A$.

14. If A and B have the same cardinality, then we have a one-to-one correspondence $f : A \to B$. The function f meets the requirement of the definition that $|A| \leq |B|$, and f^{-1} meets the requirement of the definition that $|B| \leq |A|$.

16. If a set A is countable, then we can list its elements, $a_1, a_2, a_3, \ldots, a_n, \ldots$ (possibly ending after a finite number of terms). Every subset of A consists of some (or none or all) of the items in this sequence, and we can list them in the same order in which they appear in the sequence. This gives us a sequence (again, infinite or finite) listing all the elements of the subset. Thus the subset is also countable.

18. The hypothesis gives us a one-to-one and onto function f from A to B. By Exercise 16e in the supplementary exercises for this chapter, the function S_f from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ defined by $S_f(X) = f(X)$ for all $X \subseteq A$ is one-to-one and onto. Therefore $\mathcal{P}(A)$ and $\mathcal{P}(B)$ have the same cardinality.

20. By definition, we have one-to-one onto functions $f : A \to B$ and $g : B \to C$. Then $g \circ f$ is a one-to-one onto function from A to C, so $|A| = |C|$.

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22. If $A = \emptyset$, then the only way for the conditions to be met are that $B = \emptyset$ as well, and we are done. So assume that A is nonempty. Let f be the given onto function from A to B, and let $g : \mathbf{Z}^+ \to A$ be an onto function that establishes the countability of A. (If A is finite rather than countably infinite, say of cardinality k, then the function g will be defined so that $g(1)$, $g(2)$, ..., $g(k)$ will list the elements of A, and $g(n) = g(1)$ for $n > k$.) We need to find an onto function from $\mathbf{Z}^+$ to B. The function $f \circ g$ does the trick, because the composition of two onto functions is onto (Exercise 33b in Section 2.3).

24. Because $|A| < |\mathbf{Z}^+|$, there is a one-to-one function $f : A \to \mathbf{Z}^+$. We are also given that A is infinite, so the range of f has to be infinite. We will construct a bijection g from $\mathbf{Z}^+$ to A. For each $n \in \mathbf{Z}^+$, let m be the n^{th} smallest element in the range of f. Then $g(n) = f^{-1}(m)$. The existence of g contradicts the definition of $|A| < |\mathbf{Z}^+|$, and our proof is complete.

26. We can label the rational numbers with strings from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, /, -\}$ by writing down the string that represents that rational number in its simplest form (no leading 0's, denominator not 0, no common factors greater than 1 between numerator and denominator, and the minus sign in front if the number is negative). The labels are unique. It follows immediately from Exercise 25 that the set of rational numbers is countable.

28. We can think of $\mathbf{Z}^+ \times \mathbf{Z}^+$ as the countable union of countable sets, where the i^{th} set in the collection, for $i \in \mathbf{Z}^+$, is $\{(i, n) \mid n \in \mathbf{Z}^+\}$. The statement now follows from Exercise 27.

30. There are at most two real solutions of each quadratic equation, so the number of solutions is countable as long as the number of triples (a, b, c), with a, b, and c integers, is countable. But this follows from Exercise 27 in the following way. There are a countable number of pairs (b, c), since for each b (and there are countably many b's) there are only a countable number of pairs with that b as its first coordinate. Now for each a (and there are countably many a's) there are only a countable number of triples with that a as its first coordinate (since we just showed that there are only a countable number of pairs (b, c)). Thus again by Exercise 27 there are only countably many triples.

32. We saw in Exercise 31 that

$$f(m, n) = \frac{(m + n - 2)(m + n - 1)}{2} + m$$

is a one-to-one function with domain $\mathbf{Z}^+ \times \mathbf{Z}^+$. We want to expand the domain to be $\mathbf{Z} \times \mathbf{Z}$, so things need to be spread out a little if we are to keep it one-to-one. If we can find a one-to-one function g from $\mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Z}^+ \times \mathbf{Z}^+$, then composing these two functions will be our desired one-to-one function from $\mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Z}$ (we know from Exercise 33a in Section 2.3 that the composition of one-to-one functions is one-to-one). The function suggested here is $g(m, n) = ((3m + 1)^2, (3n + 1)^2)$, so that the composed function is $(f \circ g)(m, n) = ((3m + 1)^2 + (3n + 1)^2 - 2)((3m + 1)^2 + (3n + 1)^2 - 1)/2 + (3m + 1)^2$. To see that g is one-to-one, first note that it is enough to show that the behavior in each coordinate is one-to-one; that is, the function that sends integer k to positive integer $(3k + 1)^2$ is one-to-one. To see this, first note that if $k_1 \neq k_2$ and k_1 and k_2 are both positive or both negative, then $(3k_1 + 1)^2 \neq (3k_2 + 1)^2$. And if one is nonnegative and the other is negative, then they cannot have the same images under this function because the nonnegative integers are sent to squares of numbers that leave a remainder of 1 when divided by 3 ($0 \to 1^2$, $1 \to 4^2$, $2 \to 7^2$, ...), but negative integers are sent to squares of numbers that leave a remainder of 2 when divided by 3 ($-1 \to 2^2$, $-2 \to 5^2$, $-3 \to 8^2$, ...).

34. **a)** The derivative with respect to x of $f(x) = \frac{2x-1}{2x(1-x)}$ is $f'(x) = \frac{x^2+(1-x)^2}{2x^2(1-x)^2}$, which is always positive, so f is a strictly increasing function and therefore one-to-one. It is also clear that the limit of $f(x)$ as x approaches 1 from below is ∞, and the limit of $f(x)$ as x approaches 0 from above is $-\infty$, which shows that f is onto.

b) The function $f(x) = x$ is an injection from $(0, 1)$ to $\mathbf{R}$, and the function $\frac{\arctan(x)+3}{6}$ is an injection from $\mathbf{R}$ to $(0, 1)$. It then follows from the Schröder-Bernstein theorem that $|(0, 1)| = |\mathbf{R}|$.

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36. We can encode subsets of the set of positive integers as strings of, say, 5's and 6's, where the i^{th} symbol is a 5 if i is in the subset and a 6 otherwise. If we interpret this string as a real number by putting a 0 and a decimal point in front, then we have constructed a one-to-one function from $\mathcal{P}(\mathbf{Z}^+)$ to $(0,1)$. Also, we can construct a one-to-one function from $(0,1)$ to $\mathcal{P}(\mathbf{Z}^+)$ by sending the number whose binary expansion is $0.d_1d_2d_3\ldots$ to the set $\{i \mid d_i = 1\}$. Therefore by the Schröder-Bernstein theorem we have $|\mathcal{P}(\mathbf{Z}^+)| = |(0,1)|$. By Exercise 34, $|(0,1)| = |\mathbf{R}|$, so we have shown that $|\mathcal{P}(\mathbf{Z}^+)| = |\mathbf{R}|$. (We already know from Cantor's diagonal argument that $\aleph_0 < |\mathbf{R}|$.) There is one technical point here. In order for our function from $(0,1)$ to $\mathcal{P}(\mathbf{Z}^+)$ to be well-defined, we must choose which of two equivalent expressions to represent numbers that have terminating binary expansions to use (for example, $0.10010\overline{1}$ versus $0.10011\overline{0}$); we can decide to always use the terminating form, i.e., the one ending in all 0's.)

38. We know from Example 5 that the set of real numbers between 0 and 1 is uncountable. Let us associate to each real number in this range (including 0 but excluding 1) a function from the set of positive integers to the set $\{0,1,2,3,4,5,6,7,8,9\}$ as follows: If x is a real number whose decimal representation is $0.d_1d_2d_3\ldots$ (with ambiguity resolved by forbidding the decimal to end with an infinite string of 9's), then we associate to x the function whose rule is given by $f(n) = d_n$. Clearly this is a one-to-one function from the set of real numbers between 0 and 1 and a subset of the set of all functions from the set of positive integers to the set $\{0,1,2,3,4,5,6,7,8,9\}$. Two different real numbers must have different decimal representations, so the corresponding functions are different. (A few functions are left out, because of forbidding representations such as $0.239999\ldots$.) Since the set of real numbers between 0 and 1 is uncountable, the subset of functions we have associated with them must be uncountable. But the set of all such functions has at least this cardinality, so it, too, must be uncountable (by Exercise 15).

40. We follow the hint. Suppose that f is a function from S to $\mathcal{P}(S)$. We must show that f is not onto. Let $T = \{\, s \in S \mid s \notin f(s) \,\}$. We will show that T is not in the range of f. If it were, then we would have $f(t) = T$ for some $t \in S$. Now suppose that $t \in T$. Then because $t \in f(t)$, it follows from the definition of T that $t \notin T$; this is a contradiction. On the other hand, suppose that $t \notin T$. Then because $t \notin f(t)$, it follows from the definition of T that $t \in T$; this is again a contradiction. This completes our proof by contradiction that f is not onto. On the other hand, the function sending x to $\{x\}$ for each $x \in S$ is a one-to-one function from S to $\mathcal{P}(S)$, so by Definition 2 $|S| \le |\mathcal{P}(S)|$. By the same definition, since $|S| = |\mathcal{P}(S)|$ (from what we have just proved and Definition 1), it follows that $|S| < |\mathcal{P}(S)|$.

SECTION 2.6 Matrices

2. We just add entry by entry.

a) $\begin{bmatrix} 0 & 3 & 9 \\ 1 & 4 & -1 \\ 2 & -5 & -3 \end{bmatrix}$ **b)** $\begin{bmatrix} -4 & 9 & 2 & 10 \\ -4 & -5 & 4 & 0 \end{bmatrix}$

4. To multiply matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, we compute the $(i,j)^{\text{th}}$ entry of the product $\boldsymbol{AB}$ by adding all the products of elements from the i^{th} row of $\boldsymbol{A}$ with the corresponding element in the j^{th} column of $\boldsymbol{B}$, that is $\sum_{k=1}^{n} a_{ik}b_{kj}$. This can only be done, of course, when the number of columns of $\boldsymbol{A}$ equals the number of rows of $\boldsymbol{B}$ (called n in the formula shown here).

a) $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 4 & -1 & -7 & 6 \\ -7 & -5 & 8 & 5 \\ 4 & 0 & 7 & 3 \end{bmatrix}$ **c)** $\begin{bmatrix} 2 & 0 & -3 & -4 & -1 \\ 24 & -7 & 20 & 29 & 2 \\ -10 & 4 & -17 & -24 & -3 \end{bmatrix}$

6. First note that $\boldsymbol{A}$ must be a 3×3 matrix in order for the sizes to work out as shown. If we name the elements of $\boldsymbol{A}$ in the usual way as $[a_{ij}]$, then the given equation is really nine equations in the nine unknowns a_{ij},

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obtained simply by writing down what the matrix multiplication on the left means:

$$\begin{aligned}
1 \cdot a_{11} + 3 \cdot a_{21} + 2 \cdot a_{31} &= 7 \\
1 \cdot a_{12} + 3 \cdot a_{22} + 2 \cdot a_{32} &= 1 \\
1 \cdot a_{13} + 3 \cdot a_{23} + 2 \cdot a_{33} &= 3 \\
2 \cdot a_{11} + 1 \cdot a_{21} + 1 \cdot a_{31} &= 1 \\
2 \cdot a_{12} + 1 \cdot a_{22} + 1 \cdot a_{32} &= 0 \\
2 \cdot a_{13} + 1 \cdot a_{23} + 1 \cdot a_{33} &= 3 \\
4 \cdot a_{11} + 0 \cdot a_{21} + 3 \cdot a_{31} &= -1 \\
4 \cdot a_{12} + 0 \cdot a_{22} + 3 \cdot a_{32} &= -3 \\
4 \cdot a_{13} + 0 \cdot a_{23} + 3 \cdot a_{33} &= 7
\end{aligned}$$

This is really not as bad as it looks, since each variable only appears in three equations. For example, the first, fourth, and seventh equations are a system of three equations in the three variables a_{11}, a_{21}, and a_{31}. We can solve them using standard algebraic techniques to obtain $a_{11} = -1$, $a_{21} = 2$ and $a_{31} = 1$. By similar reasoning we also obtain $a_{12} = 0$, $a_{22} = 1$ and $a_{32} = -1$; and $a_{13} = 1$, $a_{23} = 0$ and $a_{33} = 1$. Thus our answer is

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

As a check we can carry out the matrix multiplication and verify that we obtain the given right-hand side.

8. Since the entries of $\boldsymbol{A}+\boldsymbol{B}$ are $a_{ij}+b_{ij}$ and the entries of $\boldsymbol{B}+\boldsymbol{A}$ are $b_{ij}+a_{ij}$, that $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$ follows from the commutativity of addition of real numbers.

10. **a)** This product is a 3×5 matrix.

b) This is not defined since the number of columns of $\boldsymbol{B}$ does not equal the number of rows of $\boldsymbol{A}$.

c) This product is a 3×4 matrix.

d) This is not defined since the number of columns of $\boldsymbol{C}$ does not equal the number of rows of $\boldsymbol{A}$.

e) This is not defined since the number of columns of $\boldsymbol{B}$ does not equal the number of rows of $\boldsymbol{C}$.

f) This product is a 4×5 matrix.

12. We use the definition of matrix addition and multiplication. All summations here are from 1 to k.

a) $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \left[\sum (a_{iq}+b_{iq})c_{qj}\right] = \left[\sum a_{iq}c_{qj} + \sum b_{iq}c_{qj}\right] = \boldsymbol{AC}+\boldsymbol{BC}$

b) $\boldsymbol{C}(\boldsymbol{A}+\boldsymbol{B}) = \left[\sum c_{iq}(a_{qj}+b_{qj})\right] = \left[\sum c_{iq}a_{qj} + \sum c_{iq}b_{qj}\right] = \boldsymbol{CA}+\boldsymbol{CB}$

14. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two diagonal $n \times n$ matrices. Let $\boldsymbol{C} = [c_{ij}]$ be the product $\boldsymbol{AB}$. From the definition of matrix multiplication, $c_{ij} = \sum a_{iq}b_{qj}$. Now all the terms a_{iq} in this expression are 0 except for $q = i$, so $c_{ij} = a_{ii}b_{ij}$. But $b_{ij} = 0$ unless $i = j$, so the only nonzero entries of $\boldsymbol{C}$ are the diagonal entries $c_{ii} = a_{ii}b_{ii}$.

16. The $(i,j)^{\text{th}}$ entry of $(\boldsymbol{A}^t)^t$ is the $(j,i)^{\text{th}}$ entry of $\boldsymbol{A}^t$, which is the $(i,j)^{\text{th}}$ entry of $\boldsymbol{A}$.

18. We need to multiply these two matrices together in both directions and check that both products are $\boldsymbol{I}_3$. Indeed, they are.

20. **a)** Using Exercise 19, noting that $ad - bc = -5$, we write down the inverse immediately:

$$\begin{bmatrix} -3/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix}.$$

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b) We multiply to obtain $\boldsymbol{A}^2 = \begin{bmatrix} 3 & 4 \\ 2 & 11 \end{bmatrix}$ and then $\boldsymbol{A}^3 = \begin{bmatrix} 1 & 18 \\ 9 & 37 \end{bmatrix}$.

c) We multiply to obtain $(\boldsymbol{A}^{-1})^2 = \begin{bmatrix} 11/25 & -4/25 \\ -2/25 & 3/25 \end{bmatrix}$ and then $(\boldsymbol{A}^{-1})^3 = \begin{bmatrix} -37/125 & 18/125 \\ 9/125 & -1/125 \end{bmatrix}$.

d) Applying the method of Exercise 19 for obtaining inverses to the answer in part (b), we obtain the answer in part (c). Therefore $(\boldsymbol{A}^3)^{-1} = (\boldsymbol{A}^{-1})^3$.

22. A matrix is symmetric if and only if it equals its transpose. So let us compute the transpose of $\boldsymbol{A}\boldsymbol{A}^t$ and see if we get this matrix back. Using Exercise 17b and then Exercise 16, we have $(\boldsymbol{A}\boldsymbol{A}^t)^t = \big((\boldsymbol{A}^t)^t\big)\boldsymbol{A}^t = \boldsymbol{A}\boldsymbol{A}^t$, as desired.

24. a) We simply note that under the given definitions of $\boldsymbol{A}$, $\boldsymbol{X}$, and $\boldsymbol{B}$, the definition of matrix multiplication is exactly the system of equations shown.

b) The given system is the matrix equation $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{B}$. If $\boldsymbol{A}$ is invertible with inverse $\boldsymbol{A}^{-1}$, then we can multiply both sides of this equation by $\boldsymbol{A}^{-1}$ to obtain $\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$. The left-hand side simplifies to $\boldsymbol{I}\boldsymbol{X}$, however, by the definition of inverse, and this is simply $\boldsymbol{X}$. Thus the given system is equivalent to the system $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$, which obviously tells us exactly what $\boldsymbol{X}$ is (and therefore what all the values x_i are).

26. We follow the definitions.

a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

28. We follow the definition and obtain $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

30. a) $\boldsymbol{A} \vee \boldsymbol{A} = [a_{ij} \vee a_{ij}] = [a_{ij}] = \boldsymbol{A}$ **b)** $\boldsymbol{A} \wedge \boldsymbol{A} = [a_{ij} \wedge a_{ij}] = [a_{ij}] = \boldsymbol{A}$

32. a) $(\boldsymbol{A} \vee \boldsymbol{B}) \vee \boldsymbol{C} = [(a_{ij} \vee b_{ij}) \vee c_{ij}] = [a_{ij} \vee (b_{ij} \vee c_{ij})] = \boldsymbol{A} \vee (\boldsymbol{B} \vee \boldsymbol{C})$

b) This is identical to part (a), with $\wedge$ replacing $\vee$.

34. Since the i^{th} row of $\boldsymbol{I}$ consists of all 0's except for a 1 in the $(i,i)^{\text{th}}$ position, we have $\boldsymbol{I} \odot \boldsymbol{A} = [(0 \wedge a_{1j}) \vee \cdots \vee (1 \wedge a_{ij}) \vee \cdots \vee (0 \wedge a_{nj})] = [a_{ij}] = \boldsymbol{A}$. Similarly, since the j^{th} column of $\boldsymbol{I}$ consists of all 0's except for a 1 in the $(j,j)^{\text{th}}$ position, we have $\boldsymbol{A} \odot \boldsymbol{I} = [(a_{i1} \wedge 0) \vee \cdots \vee (a_{ij} \wedge 1) \vee \cdots \vee (a_{in} \wedge 0)] = [a_{ij}] = \boldsymbol{A}$.

SUPPLEMENTARY EXERCISES FOR CHAPTER 2

2. We are given that $A \subseteq B$. We want to prove that the power set of A is a subset of the power set of B, which means that if $C \subseteq A$ then $C \subseteq B$. But this follows directly from Exercise 17 in Section 2.1.

4. a) $\mathbf{Z}$ **b)** $\emptyset$ **c)** $\mathbf{O}$ **d)** $\mathbf{E}$

6. If $A \subseteq B$, then every element in A is also in B, so clearly $A \cap B = A$. Conversely, if $A \cap B = A$, then every element of A must also be in $A \cap B$, and hence in B. Therefore $A \subseteq B$.

8. This identity is true, so we must show that every element in the left-hand side is also an element in the right-hand side and conversely. Let $x \in (A - B) - C$. Then $x \in A - B$ but $x \notin C$. This means that $x \in A$, but $x \notin B$ and $x \notin C$. Therefore $x \in A - C$, and therefore $x \in (A - C) - B$. The converse is proved in exactly the same way.

10. The inequality follows from the obvious fact that $A \cap B \subseteq A \cup B$. Equality can hold only if there are no elements in either A or B that are not in both A and B, and this can happen only if $A = B$.

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12. Since $\overline{A} \cap \overline{B} = \overline{(A \cup B)}$, we are asked to show that $|\overline{(A \cup B)}| = |U| - \big(|A| + |B| - |A \cap B|\big)$. This follows immediately from the facts that $|\overline{X}| = |U| - |X|$ (which is clear from the definitions) and (see the discussion following Example 5 in Section 2.2) that $|A \cup B| = |A| + |B| - |A \cap B|$.

14. Define a function $g : f(S) \to S$ by choosing, for each element x in $f(S)$, an element $g(x) \in S$ such that $f(g(x)) = x$. Clearly g is one-to-one, so $|f(S)| \le |S|$. Note that we do not need the hypothesis that A and B are finite.

16. **a)** We are given that f is one-to-one, and we must show that S_f is one-to-one. So suppose that $X_1 \neq X_2$, where these are subsets of A. We have to show that $S_f(X_1) \neq S_f(X_2)$. Without loss of generality there is an element $a \in X_1 - X_2$. This means that $f(a) \in S_f(X_1)$. If $f(a)$ were also an element of $S_f(X_2)$, then we would need an element $a' \in X_2$ such that $f(a') = f(a)$. But since f is one-to-one, this forces $a' = a$, which is impossible, because $a \notin X_2$. Therefore $f(a) \in S_f(X_1) - S_f(X_2)$, so $S_f(X_1) \neq S_f(X_2)$.

b) We are given that f is onto, and we must show that S_f is onto. So suppose that $Y \subseteq B$. We have to find $X \subseteq A$ such that $S_f(X) = Y$. Let $X = \{\, x \in A \mid f(x) \in Y \,\}$. We claim that $S_f(X) = Y$. Clearly $S_f(X) \subseteq Y$. To see that $Y \subseteq S_f(X)$, suppose that $b \in Y$. Then because f is onto, there is some $a \in A$ such that $f(a) = b$. By our definition of X, $a \in X$. Therefore by definition $b \in S_f(X)$.

c) We are given that f is onto, and we must show that $S_{f^{-1}}$ is one-to-one. So suppose that $Y_1 \neq Y_2$, where these are subsets of B. We have to show that $S_{f^{-1}}(Y_1) \neq S_{f^{-1}}(Y_2)$. Without loss of generality there is an element $b \in Y_1 - Y_2$. Because f is onto, there is an $a \in A$ such that $f(a) = b$. Therefore $a \in S_{f^{-1}}(Y_1)$. But we also know that $a \notin S_{f^{-1}}(Y_2)$, because if a were an element of $S_{f^{-1}}(Y_2)$, then we would have $b = f(a) \in Y_2$, contrary to our choice of b. The existence of this a shows that $S_{f^{-1}}(Y_1) \neq S_{f^{-1}}(Y_2)$.

d) We are given that f is one-to-one, and we must show that $S_{f^{-1}}$ is onto. So suppose that $X \subseteq A$. We have to find $Y \subseteq B$ such that $S_{f^{-1}}(Y) = X$. Let $Y = S_f(X)$. In other words, $Y = \{\, f(x) \mid x \in X \,\}$. We must show that $S_{f^{-1}}(Y) = X$, which means that we must show that $\{\, u \in A \mid f(u) \in \{\, f(x) \mid x \in X \,\} \,\} = X$ (we changed the dummy variable to u for clarity). That the right-hand side is a subset of the left-hand side is immediate, because if $u \in X$, then $f(u)$ is an $f(x)$ for some $x \in X$. Conversely, suppose that u is in the left-hand side. Thus $f(u) = f(x_0)$ for some $x_0 \in X$. But because f is one-to-one, we know that $u = x_0$; that is $u \in X$.

e) This follows immediately from the earlier parts, because to be a one-to-one correspondence means to be one-to-one and onto.

18. If n is even , then $n/2$ is an integer, so $\lceil n/2 \rceil + \lfloor n/2 \rfloor = (n/2) + (n/2) = n$. If n is odd, then $\lceil n/2 \rceil = (n+1)/2$ and $\lfloor n/2 \rfloor = (n-1)/2$, so again the sum is n.

20. This is certainly true if either x or y is an integer, since then this equation is equivalent to the identity (4b) in Table 1 of Section 2.3. Otherwise, write x and y in terms of their integer and fractional parts: $x = n + \epsilon$ and $y = m + \delta$, where $n = \lfloor x \rfloor$, $0 < \epsilon < 1$, $m = \lfloor y \rfloor$, and $0 < \delta < 1$. If $\delta + \epsilon > 1$, then the equation is true, since both sides equal $m + n + 2$; if $\delta + \epsilon \le 1$, then the equation is false, since the left-hand side equals $m + n + 1$, but the right-hand side equals $m + n + 2$. To summarize: the equation is true if and only if either at least one of x and y is an integer or the sum of the fractional parts of x and y exceeds 1.

22. The values of the floor and ceiling function will depend on whether their arguments are integral or not. So there seem to be two cases here. First let us suppose that n is even. Then $n/2$ is an integer, and $n^2/4$ is also an integer, so the equation is a simple algebraic fact. The second case is harder. Suppose that n is odd, say $n = 2k + 1$. Then $n/2 = k + \frac{1}{2}$. Therefore the left-hand side gives us $k(k+1) = k^2 + k$, since we have to round down for the first factor and round up for the second. What about the right-hand side? $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$, so $n^2/4 = k^2 + k + \frac{1}{4}$. Therefore the floor function gives us $k^2 + k$, and the proof is completed.

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24. Since we are dividing by 4, let us write $x = 4n - k$, where $0 \le k < 4$. In other words, write x in terms of how much it is less than the smallest multiple of 4 not less than it. There are three cases. If $k = 0$, then x is already a multiple of 4, so both sides equal n. If $0 < k \le 2$, then $\lfloor x/2 \rfloor = 2n - 1$, so the left-hand side is $\lfloor n - \frac{1}{2} \rfloor = n - 1$. Of course the right-hand side is $n - 1$ as well, so again the two sides agree. Finally, suppose that $2 < k < 4$. Then $\lfloor x/2 \rfloor = 2n - 2$, and the left-hand side is $\lfloor n - 1 \rfloor = n - 1$; of course the right-hand side is still $n - 1$, as well. Since we proved that the two sides are equal in all cases, the proof is complete.

26. If x is an integer, then of course the two sides are identical. So suppose that $x = k + \epsilon$, where k is an integer and ϵ is a real number with $0 < \epsilon < 1$. Then the values of the left-hand side, which is $\lfloor (k + n)/m \rfloor$, and the right-hand side, which is $\lfloor (k + n + \epsilon)/m \rfloor$, are the same, since adding a number strictly between 0 and 1 to the numerator of a fraction whose numerator and denominator are integers cannot cause the fraction to reach the next higher integer value (the numerator cannot reach the next multiple of m).

28. **a)** $1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69$

b) Suppose there were only a finite set of Ulam numbers, say $u_1 < u_2 < \cdots < u_n$. Then it is clear that $u_{n-1} + u_n$ can be written uniquely as the sum of two distinct Ulam numbers, so this is an Ulam number larger than u_n, a contradiction. Therefore there are an infinite number of Ulam numbers.

30. If we work at this long enough, we might notice that each term after the first three is the sum of the previous three terms. With this rule the next four terms will be 169, 311, 572, 1052. One way to use the power of technology here is to submit the given sequence to The On-Line Encyclopedia of Integer Sequences (`oeis.org`).

32. We know that the set of rational numbers is countable. If the set of irrational numbers were also countable, then the union of these two sets would also be countable by Theorem 1 in Section 2.5. But their union, the set of real numbers, is known to be uncountable. This contradiction tells us that the set of irrational numbers is not countable.

34. A finite subset of $\mathbf{Z}^+$ has a largest element and therefore is a subset of $\{1, 2, 3, \ldots, n\}$ for some positive integer n. Let S_n be the set of subsets of $\{1, 2, 3, \ldots, n\}$. It is finite and therefore countable; in fact $|S_n| = 2^n$. The set of all finite subsets of $\mathbf{Z}^+$ is the union $\bigcup_{n=1}^{\infty} S_n$. Being a countable union of countable sets, it is countable by Exercise 27 in Section 2.5.

36. This follows immediately from Exercise 35, because $\mathbf{C}$ can be identified with $\mathbf{R} \times \mathbf{R}$ by sending the complex number $a + bi$, where a and b are real numbers, to the ordered pair (a, b).

38. Since $\boldsymbol{A}$ is the matrix defined by $a_{ii} = c$ and $a_{ij} = 0$ for $i \ne j$, it is easy to see from the definition of multiplication that $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are both the same as $\boldsymbol{B}$ except that every entry has been multiplied by c. Therefore these two matrices are equal.

40. We simply need to show that the alleged inverse of $\boldsymbol{AB}$ has the correct defining property—that its product with $\boldsymbol{AB}$ (on either side) is the identity. Thus we compute

$$(\boldsymbol{AB})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{B}^{-1})\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I},$$

and similarly $(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{AB}) = \boldsymbol{I}$. Therefore $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. (Note that the indicated matrix multiplications were all defined, since the hypotheses implied that both $\boldsymbol{A}$ and $\boldsymbol{B}$ were $n \times n$ matrices for some (and the same) n.)

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