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SOLUTIONS TO CHAPTER 2

Problem 2.1

(a) The firm's problem is to choose the quantities of capital, K, and effective labor, AL, to minimize costs, wAL + rK, subject to the production function, Y = ALf(k). Set up the Lagrangian:

(1) $\mathcal{L} = wAL + rK + \lambda\left[Y - ALf(K/AL)\right]$.

The first-order condition for K is given by

(2) $\dfrac{\partial \mathcal{L}}{\partial K} = r - \lambda\left[ALf'(K/AL)(1/AL)\right] = 0$,

which implies that

(3) $r = \lambda f'(k)$.

The first-order condition for AL is given by

(4) $\dfrac{\partial \mathcal{L}}{\partial (AL)} = w - \lambda\left[f(K/AL) + ALf'(K/AL)(-K)/(AL)^2\right] = 0$,

which implies that

(5) $w = \lambda\left[f(k) - kf'(k)\right]$.

Dividing equation (3) by equation (5) gives us

(6) $\dfrac{r}{w} = \dfrac{f'(k)}{f(k) - kf'(k)}$.

Equation (6) implicitly defines the cost-minimizing choice of k. Clearly this choice does not depend upon the level of output, Y. Note that equation (6) is the standard cost-minimizing condition: the ratio of the marginal cost of the two inputs, capital and effective labor, must equal the ratio of the marginal products of the two inputs.

(b) Since, as shown in part (a), each firm chooses the same value of k and since we are told that each firm has the same value of A, we can write the total amount produced by the N cost-minimizing firms as

(7) $\sum_{i=1}^{N} Y_i = \sum_{i=1}^{N} AL_i f(k) = Af(k)\sum_{i=1}^{N} L_i = A\bar{L}f(k)$,

where $\bar{L}$ is the total amount of labor employed.

The single firm also has the same value of A and would choose the same value of k; the choice of k does not depend on Y. Thus if it used all of the labor employed by the N cost-minimizing firms, $\bar{L}$, the single firm would produce $Y = A\bar{L}f(k)$. This is the same amount of output produced in total by the N cost-minimizing firms.

Problem 2.2

(a) The individual's problem is to maximize lifetime utility given by

(1) $U = \dfrac{C_1^{1-\theta}}{1-\theta} + \dfrac{1}{1+\rho}\dfrac{C_2^{1-\theta}}{1-\theta}$,

subject to the lifetime budget constraint given by

(2) $P_1C_1 + P_2C_2 = W$,

where W represents lifetime income.

Rearrange the budget constraint to solve for C_2 :

(3) $C_2 = W/P_2 - C_1P_1/P_2$.

Substitute equation (3) into equation (1):

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(4) $U = \dfrac{C_1^{1-\theta}}{1-\theta} + \dfrac{1}{1+\rho}\dfrac{\left[W/P_2 - C_1 P_1/P_2\right]^{1-\theta}}{1-\theta}$.

Now we can solve the unconstrained problem of maximizing utility, as given by equation (4), with respect to first period consumption, C_1. The first-order condition is given by

(5) $\partial U/\partial C_1 = C_1^{-\theta} + (1/1+\rho)C_2^{-\theta}(-P_1/P_2) = 0$.

Solving for C_1 gives us

(6) $C_1^{-\theta} = (1/1+\rho)(P_1/P_2)C_2^{-\theta}$,

or simply

(7) $C_1 = (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}C_2$.

In order to solve for C_2, substitute equation (7) into equation (3):

(8) $C_2 = W/P_2 - (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}C_2(P_1/P_2)$.

Simplifying yields

(9) $C_2[1+(1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}] = W/P_2$,

or simply

(10) $C_2 = \dfrac{W/P_2}{\left[1+(1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}\right]}$.

Finally, to get the optimal choice of C_1, substitute equation (10) into equation (7):

(11) $C_1 = \dfrac{(1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}(W/P_2)}{\left[1+(1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}\right]}$.

(b) From equation (7), the optimal ratio of first-period to second-period consumption is

(12) $C_1/C_2 = (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}$.

Taking the natural logarithm of both sides of equation (12) yields

(13) $\ln(C_1/C_2) = (1/\theta)\ln(1+\rho) + (1/\theta)\ln(P_2/P_1)$.

The elasticity of substitution between C_1 and C_2, defined in such a way that it is positive, is given by

(14) $\dfrac{\partial(C_1/C_2)}{\partial(P_2/P_1)}\dfrac{(P_2/P_1)}{(C_1/C_2)} = \dfrac{\partial[\ln(C_1/C_2)]}{\partial[\ln(P_2/P_1)]} = \dfrac{1}{\theta}$,

where we have used equation (13) to find the derivative. Thus higher values of θ imply that the individual is less willing to substitute consumption between periods.

Problem 2.3

(a) We can use analysis similar to the intuitive derivation of the Euler equation in Section 2.2 of the text. Think of the household's consumption at two moments of time. Specifically, consider a short (formally infinitesimal) period of time Δt from $(t_0 - \varepsilon)$ to $(t_0 + \varepsilon)$.

Imagine the household reducing consumption per unit of effective labor, c, at $(t_0 - \varepsilon)$ – an instant before the confiscation of wealth – by a small (again, infinitesimal) amount Δc. It then invests this additional saving and consumes the proceeds at $(t_0 + \varepsilon)$. If the household is optimizing, the marginal impact of this change on lifetime utility must be zero.

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This experiment would have a utility cost of $u'(c_{before})\Delta c$. Ordinarily, since the instantaneous rate of return is r(t), c at time $(t_0 + \varepsilon)$ could be increased by $e^{[r(t)-n-g]\Delta t}\Delta c$. But here, half of that increase will be confiscated. Thus the utility benefit would be $[1/2]u'(c_{after})e^{[r(t)-n-g]\Delta t}\Delta c$. Thus for the path of consumption to be utility-maximizing, it must satisfy

(1) $u'(c_{before})\Delta c = \frac{1}{2}u'(c_{after})e^{[r(t)-n-g]\Delta t}\Delta c.$

Rather informally, we can cancel the Δc's and allow $\Delta t \to 0$, leaving us with

(2) $u'(c_{before}) = \frac{1}{2}u'(c_{after}).$

Thus there will be a discontinuous jump in consumption at the time of the confiscation of wealth. Specifically, consumption will jump down. Intuitively, the household's consumption will be high before t_0 because it will have an incentive not to save to avoid the wealth confiscation.

(b) In this case, from the viewpoint of an individual household, its actions will not affect the amount of wealth that is confiscated. For an individual household, essentially a predetermined amount of wealth will be confiscated at time t_0 and thus the household's optimization and its choice of consumption path would take this into account. The household would still prefer to smooth consumption over time and there will not be a discontinuous jump in consumption at time t_0.

Problem 2.4

We need to solve the household's problem assuming log utility and in per capita terms rather than in units of effective labor. The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize

(1) $U = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t) \frac{L(t)}{H} dt,$

subject to

(2) $\int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt = \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} A(t) w(t) \frac{L(t)}{H} dt.$

Now let $W \equiv \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} A(t) w(t) \frac{L(t)}{H} dt.$

We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

(3) $\mathcal{L} = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t) \frac{L(t)}{H} dt + \lambda \left[W - \int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \right].$

The first-order condition is given by

(4) $\frac{\partial \mathcal{L}}{\partial C(t)} = e^{-\rho t} C(t)^{-1} \frac{L(t)}{H} - \lambda e^{-R(t)} \frac{L(t)}{H} = 0.$

Canceling the L(t)/H yields

(5) $e^{-\rho t} C(t)^{-1} = \lambda e^{-R(t)},$

which implies

(6) $C(t) = e^{-\rho t} \lambda^{-1} e^{R(t)}.$

Substituting equation (6) into the budget constraint given by equation (2) leaves us with

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(7) $\int_{t=0}^{\infty} e^{-R(t)}\left[e^{-\rho t}\lambda^{-1}e^{R(t)}\right]\frac{L(t)}{H}dt = W.$

Since $L(t) = e^{nt} L(0)$, this implies

(8) $\lambda^{-1}\frac{L(0)}{H}\int_{t=0}^{\infty} e^{-(\rho-n)t}dt = W.$

As long as $\rho - n > 0$ (which it must be), the integral is equal to $1/(\rho - n)$ and thus λ^{-1} is given by

(9) $\lambda^{-1} = \frac{W}{L(0)/H}(\rho - n).$

Substituting equation (9) into equation (6) yields

(10) $C(t) = e^{R(t)-\rho t}\left[\frac{W}{L(0)/H}(\rho - n)\right].$

Initial consumption is therefore

(11) $C(0) = \frac{W}{L(0)/H}(\rho - n).$

Note that C(0) is consumption per person, W is wealth per household and L(0)/H is the number of people per household. Thus W/[L(0)/H] is wealth per person. This equation says that initial consumption per person is a constant fraction of initial wealth per person, and $(\rho - n)$ can be interpreted as the marginal propensity to consume out of wealth. With logarithmic utility, this propensity to consume is independent of the path of the real interest rate. Also note that the bigger is the household's discount rate ρ – the more the household discounts the future – the bigger is the fraction of wealth that it initially consumes.

Problem 2.5

The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize

(1) $U = \int_{t=0}^{\infty} e^{-\rho t}\frac{C(t)^{1-\theta}}{1-\theta}\frac{L(t)}{H}dt,$

subject to

(2) $\int_{t=0}^{\infty} e^{-rt}C(t)\frac{L(t)}{H}dt = W,$

where W denotes the household's initial wealth plus the present value of its lifetime labor income, i.e. the right-hand side of equation (2.7) in the text. Note that the real interest rate, r, is assumed to be constant.

We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

(3) $\mathcal{L} = \int_{t=0}^{\infty} e^{-\rho t}\frac{C(t)^{1-\theta}}{1-\theta}\frac{L(t)}{H}dt + \lambda\left[W - \int_{t=0}^{\infty} e^{-rt}C(t)\frac{L(t)}{H}dt\right].$

The first-order condition is given by

(4) $\frac{\partial\mathcal{L}}{\partial C(t)} = e^{-\rho t}C(t)^{-\theta}\frac{L(t)}{H} - \lambda e^{-rt}\frac{L(t)}{H} = 0.$

Canceling the L(t)/H yields

(5) $e^{-\rho t}C(t)^{-\theta} = \lambda e^{-rt}.$

Differentiate both sides of equation (5) with respect to time:

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(6) $e^{-\rho t}\left[-\theta C(t)^{-\theta-1}\dot{C}(t)\right] - \rho e^{-\rho t}C(t)^{-\theta} + r\lambda e^{-rt} = 0.$

This can be rearranged to obtain

(7) $-\theta\frac{\dot{C}(t)}{C(t)}e^{-\rho t}C(t)^{-\theta} - \rho e^{-\rho t}C(t)^{-\theta} + r\lambda e^{-rt} = 0.$

Now substitute the first-order condition, equation (5), into equation (7):

(8) $-\theta\frac{\dot{C}(t)}{C(t)}\lambda e^{-rt} - \rho\lambda e^{-rt} + r\lambda e^{-rt} = 0.$

Canceling the λe^{-rt} and solving for the growth rate of consumption, $\dot{C}(t)/C(t)$, yields

(9) $\frac{\dot{C}(t)}{C(t)} = \frac{r-\rho}{\theta}.$

Thus with a constant real interest rate, the growth rate of consumption is a constant. If $r > \rho$ – that is, if the rate that the market pays to defer consumption exceeds the household's discount rate – consumption will be rising over time. The value of θ determines the magnitude of consumption growth if r exceeds ρ. A lower value of θ – and thus a higher value of the elasticity of substitution, $1/\theta$ – means that consumption growth will be higher for any given difference between r and ρ.

We now need to solve for the path of C(t). First, note that equation (9) can be rewritten as

(10) $\frac{\partial \ln C(t)}{\partial t} = \frac{r-\rho}{\theta}.$

Integrate equation (10) forward from time $\tau = 0$ to time $\tau = t$:

(11) $\ln C(t) - \ln C(0) = \left[(r-\rho)/\theta\right]\tau\Big|_{\tau=0}^{\tau=t},$

which simplifies to

(12) $\ln\left[C(t)/C(0)\right] = \left[(r-\rho)/\theta\right]t.$

Taking the exponential function of both sides of equation (12) yields

(13) $C(t)/C(0) = e^{\left[(r-\rho)/\theta\right]t},$

and thus

(14) $C(t) = C(0)e^{\left[(r-\rho)/\theta\right]t}.$

We can now solve for initial consumption, C(0), by using the fact that it must be chosen to satisfy the household's budget constraint. Substitute equation (14) into equation (2):

(15) $\int_{t=0}^{\infty} e^{-rt}C(0)e^{\left[(r-\rho)/\theta\right]t}\frac{L(t)}{H}dt = W.$

Using the fact that $L(t) = L(0)e^{nt}$ yields

(16) $\frac{C(0)L(0)}{H}\int_{t=0}^{\infty} e^{-\left[\rho-r+\theta(r-n)\right]t/\theta}dt = W.$

As long as $[\rho - r + \theta(r - n)]/\theta > 0$, we can solve the integral:

(17) $\int_{t=0}^{\infty} e^{-\left[\rho-r+\theta(r-n)\right]t/\theta}dt = \frac{\theta}{\rho-r+\theta(r-n)}.$

Substitute equation (17) into equation (16) and solve for C(0):

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(18) $C(0) = \dfrac{W}{L(0)/H}\left[\dfrac{(\rho - r)}{\theta} + (r - n)\right]$.

Finally, to get an expression for consumption at each instant in time, substitute equation (18) into equation (14):

(19) $C(t) = e^{\left[(r-\rho)/\theta\right]t}\dfrac{W}{L(0)/H}\left[\dfrac{(\rho - r)}{\theta} + (r - n)\right]$.

Problem 2.6

(a) The equation describing the dynamics of the capital stock per unit of effective labor is

(1) $\dot{k}(t) = f\big(k(t)\big) - c(t) - (n + g)k(t)$.

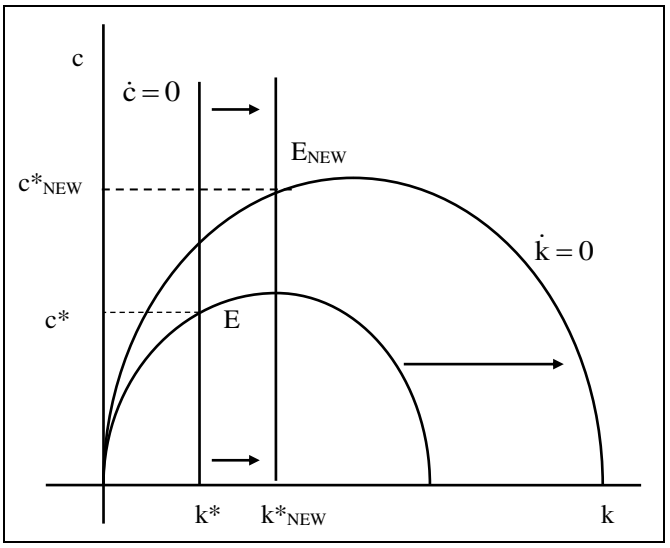
For a given k, the level of c that implies $\dot{k} = 0$ is given by c = f(k) - (n + g)k. Thus a fall in g makes the level of c consistent with $\dot{k} = 0$ higher for a given k. That is, the $\dot{k} = 0$ curve shifts up. Intuitively, a lower g makes break-even investment lower at any given k and thus allows for more resources to be devoted to consumption and still maintain a given k. Since (n + g)k falls proportionately more at higher levels of k, the $\dot{k} = 0$ curve shifts up more at higher levels of k. See the figure.

(b) The equation describing the dynamics of consumption per unit of effective labor is given by

(2) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'\big(k(t)\big) - \rho - \theta g}{\theta}$.

Thus the condition required for $\dot{c} = 0$ is given by f ' (k) = ρ + θg. After the fall in g, f ' (k) must be lower for $\dot{c} = 0$. Since f " (k) is negative this means that the k needed for $\dot{c} = 0$ therefore rises. Thus the $\dot{c} = 0$ curve shifts to the right.

(c) At the time of the change in g, the value of k, the stock of capital per unit of effective labor, is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k* on the old balanced growth path.



In contrast, c, the rate at which households are consuming in units of effective labor, can jump at the time of the shock. For the economy to reach the new balanced growth path, c must jump at the instant of the change so that the economy is on the new saddle path.

However, we cannot tell whether the new saddle path passes above or below the original point E. Thus we cannot tell whether c jumps up or down and in fact, if the new saddle path passes right through point E, c might even remain the same at the instant that g falls. Thereafter, c and k rise gradually to their new balanced-growth-path values; these are higher than their values on the original balanced growth path.

(d) As explained in the solution to part (c), at the time of the change in g, k does not change. Thus, f'(k) ≡ r does not change. So the immediate impact of the decrease in g is that (r – g) increases.

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(e) In the long run, $r = \rho + \theta g$ and so $(r - g) = \rho + (\theta - 1)g$. Thus

(3) $\dfrac{\partial(r-g)}{\partial g} = (\theta - 1)$.

Thus the long-run impact of a decrease in g on $(r - g)$ depends on θ. Specifically, if $\theta < 1$, then $\partial(r-g)/\partial g < 0$ and a decrease in g would cause $(r - g)$ to rise. In the case of logarithmic utility, with $\theta = 1$, the change in g would have no long-run impact on $(r - g)$. Finally, if $\theta > 1$, then $\partial(r-g)/\partial g > 0$ and so a decrease in g would cause $(r - g)$ to fall. Intuitively, a high value of θ means that the individual has a strong desire to smooth consumption over time. Since consumption grows at rate g on the balanced growth path, a higher value of g would require more compensation (i.e. an even higher r) to tolerate this higher consumption growth. Similarly, a decline in the growth rate g means a decline in the required compensation.

(f) On a balanced growth path, the fraction of output that is saved and invested is given by $[f(k^*) - c^*]/f(k^*)$. Since k is constant, or $\dot{k} = 0$ on a balanced growth path then, from equation (1), we can write $f(k^*) - c^* = (n + g)k^*$. Using this, we can rewrite the fraction of output that is saved on a balanced growth path as

(4) $s = [(n + g)k^*]/f(k^*)$.

Differentiating both sides of equation (4) with respect to g yields

(5) $\dfrac{\partial s}{\partial g} = \dfrac{f(k^*)\left[(n+g)(\partial k^*/\partial g) + k^*\right] - (n+g)k^* f'(k^*)(\partial k^*/\partial g)}{\left[f(k^*)\right]^2}$,

which simplifies to

(6) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)[f(k^*) - k^* f'(k^*)](\partial k^*/\partial g) + f(k^*)k^*}{\left[f(k^*)\right]^2}$.

Since k^* is defined by $f'(k^*) = \rho + \theta g$, differentiating both sides of this expression with respect to g gives us $f''(k^*)(\partial k^*/\partial g) = \theta$. Solving for $\partial k^*/\partial g$ yields

(7) $\partial k^*/\partial g = \theta/f''(k^*) < 0$.

Substituting equation (7) into equation (6) yields

(8) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)[f(k^*) - k^* f'(k^*)]\theta + f(k^*)k^* f''(k^*)}{\left[f(k^*)\right]^2 f''(k^*)}$.

The first term in the numerator is positive whereas the second is negative. Thus the sign of $\partial s/\partial g$ is ambiguous; we cannot tell whether the fall in g raises or lowers the saving rate on the new balanced growth path.

(g) When the production function is Cobb-Douglas, $f(k) = k^{\alpha}$, $f'(k) = \alpha k^{\alpha-1}$ and $f''(k) = \alpha(\alpha - 1)k^{\alpha-2}$. Substituting these facts into equation (8) yields

(9) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)[k^{*\alpha} - k^* \alpha k^{*\alpha-1}]\theta + k^{*\alpha} k^* \alpha(\alpha-1)k^{*\alpha-2}}{k^{*\alpha} k^{*\alpha} \alpha(\alpha-1)k^{*\alpha-2}}$.

Equation (9) simplifies to

(10) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)k^{*\alpha}(1-\alpha)\theta - (1-\alpha)k^{*\alpha}\alpha k^{*\alpha-1}}{[-(1-\alpha)k^{*\alpha}(\alpha k^{*\alpha-1})(\alpha k^{*\alpha-1})/\alpha]}$.

Dividing the top and bottom of the right-hand-side of equation (10) by $(1 - \alpha)k^{*\alpha}$ and using the fact that $\alpha k^{*\alpha-1}$, which is $f'(k)$, equals $(\rho + \theta g)$ on the balanced growth path gives us

(11) $\dfrac{\partial s}{\partial g} = -\alpha \dfrac{[(n+g)\theta - (\rho + \theta g)]}{(\rho + \theta g)^2}$.

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Thus, finally, we have

(12) $\dfrac{\partial s}{\partial g} = -\alpha\dfrac{(n\theta-\rho)}{(\rho+\theta g)^2} = \alpha\dfrac{(\rho-n\theta)}{(\rho+\theta g)^2}.$

Problem 2.7

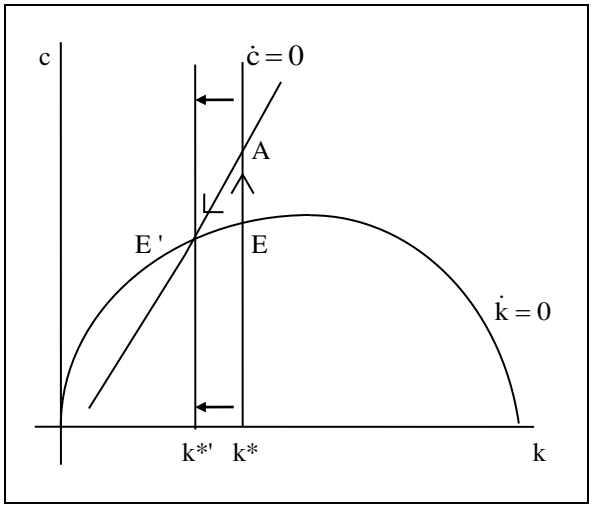
The two equations of motion are

(1) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t))-\rho-\theta g}{\theta},$

and

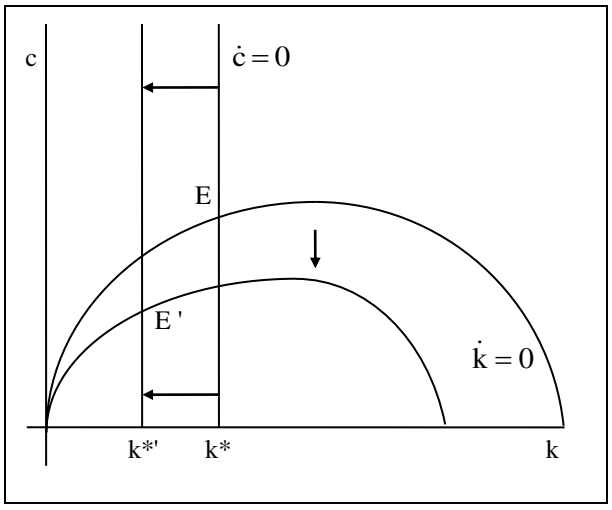
(2) $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$

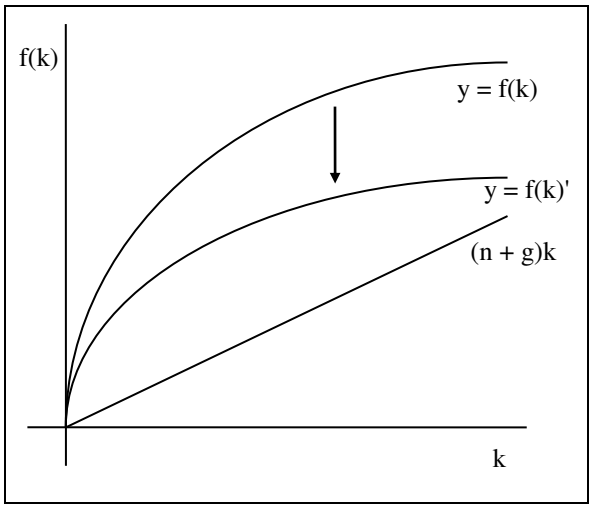
(a) A rise in θ or a fall in the elasticity of substitution, $1/\theta$, means that households become less willing to substitute consumption between periods. It also means that the marginal utility of consumption falls off more rapidly as consumption rises. If the economy is growing, this tends to make households value present consumption more than future consumption.



The capital-accumulation equation is unaffected. The condition required for $\dot{c}=0$ is given by f ' (k)= ρ + θg. Since f " (k) < 0, the f ' (k) that makes $\dot{c}=0$ is now higher. Thus the value of k that satisfies $\dot{c}=0$ is lower. The $\dot{c}=0$ locus shifts to the left. The economy moves up to point A on the new saddle path; people consume more now. Movement is then down along the new saddle path until the economy reaches point E '. At that point, c* and k* are lower than their original values.

(b) We can assume that a downward shift of the production function means that for any given k, both f(k) and f ' (k) are lower than before.



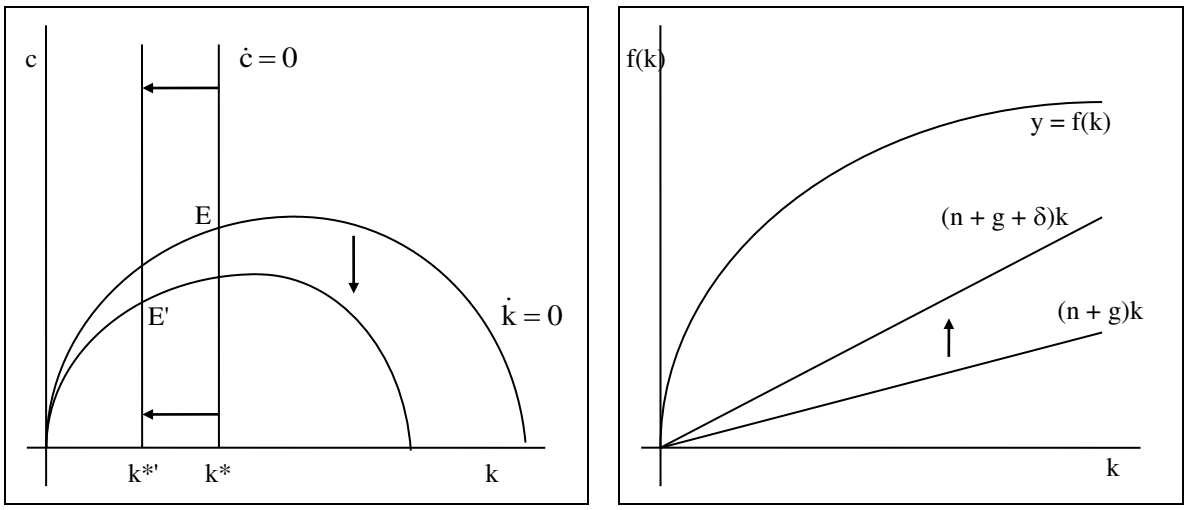


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The condition required for $\dot{k} = 0$ is given by $c = f(k) - (n + g)k$. We can see from the figure on the right that the $\dot{k} = 0$ locus will shift down more at higher levels of k. Also, since for a given k, f ' (k) is lower now, the golden-rule k will be lower than before. Thus the $\dot{k} = 0$ locus shifts as depicted in the figure.

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. For a given k, f ' (k) is now lower. Thus we need a lower k to keep f ' (k) the same and satisfy the $\dot{c} = 0$ equation. Thus the $\dot{c} = 0$ locus shifts left. The economy will eventually reach point E ' with lower c* and lower k*. Whether c initially jumps up or down depends upon whether the new saddle path passes above or below point E.

(c) With a positive rate of depreciation, $\delta > 0$, the new capital-accumulation equation is
(3) $\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t)$.



The level of saving and investment required just to keep any given k constant is now higher – and thus the amount of consumption possible is now lower – than in the case with no depreciation. The level of extra investment required is also higher at higher levels of k. Thus the $\dot{k} = 0$ locus shifts down more at higher levels of k.

In addition, the real return on capital is now f ' (k(t)) - δ and so the household's maximization will yield
(4) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$.
The condition required for $\dot{c} = 0$ is $f'(k) = \delta + \rho + \theta g$. Compared to the case with no depreciation, f ' (k) must be higher and k lower in order for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left. The economy will eventually wind up at point E ' with lower levels of c* and k*. Again, whether c jumps up or down initially depends upon whether the new saddle path passes above or below the original equilibrium point of E.

Problem 2.8

With a positive depreciation rate, $\delta > 0$, the Euler equation and the capital-accumulation equation are given by

(1) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$, and (2) $\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t)$.

We begin by taking first-order Taylor approximations to (1) and (2) around k = k* and c = c*. That is, we can write

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(3) $\dot{c} \cong \frac{\partial \dot{c}}{\partial k}[k - k^*] + \frac{\partial \dot{c}}{\partial c}[c - c^*]$, and (4) $\dot{k} \cong \frac{\partial \dot{k}}{\partial k}[k - k^*] + \frac{\partial \dot{k}}{\partial c}[c - c^*]$,

where $\partial\dot{c}/\partial k$, $\partial\dot{c}/\partial c$, $\partial\dot{k}/\partial k$ and $\partial\dot{k}/\partial c$ are all evaluated at $k = k^*$ and $c = c^*$.

Define $\tilde{c} \equiv c - c^*$ and $\tilde{k} \equiv k - k^*$. Since c^* and k^* are constants, $\dot{c}$ and $\dot{k}$ are equivalent to $\dot{\tilde{c}}$ and $\dot{\tilde{k}}$ respectively. We can therefore rewrite (3) and (4) as

(5) $\dot{\tilde{c}} \cong \frac{\partial \dot{c}}{\partial k}\tilde{k} + \frac{\partial \dot{c}}{\partial c}\tilde{c}$, and (6) $\dot{\tilde{k}} \cong \frac{\partial \dot{k}}{\partial k}\tilde{k} + \frac{\partial \dot{k}}{\partial c}\tilde{c}$.

Using equations (1) and (2) to compute these derivatives yields

(7) $\left.\frac{\partial \dot{c}}{\partial k}\right|_{bgp} = \frac{f''(k^*)c^*}{\theta}$, (8) $\left.\frac{\partial \dot{c}}{\partial c}\right|_{bgp} = \frac{f'(k^*) - \delta - \rho - \theta g}{\theta} = 0$,

(9) $\left.\frac{\partial \dot{k}}{\partial k}\right|_{bgp} = f'(k^*) - (n + g + \delta)$, (10) $\left.\frac{\partial \dot{k}}{\partial c}\right|_{bgp} = -1$.

Substituting equations (7) and (8) into (5) and equations (9) and (10) into (6) yields

(11) $\dot{\tilde{c}} \cong \frac{f''(k^*)c^*}{\theta}\tilde{k}$, and

(12) $\begin{aligned}\dot{\tilde{k}} &\cong \left[f'(k^*) - (n + g + \delta)\right]\tilde{k} - \tilde{c} \\ &\cong \left[(\delta + \rho + \theta g) - (n + g + \delta)\right]\tilde{k} - \tilde{c} \\ &\cong \beta\tilde{k} - \tilde{c}.\end{aligned}$

The second line of equation (12) uses the fact that (1) implies that $f'(k^*) = \delta + \rho + \theta g$. The third line uses the definition of $\beta \equiv \rho - n - (1 - \theta)g$.

Dividing equation (11) by $\tilde{c}$ and dividing equation (12) by $\tilde{k}$ yields

(13) $\frac{\dot{\tilde{c}}}{\tilde{c}} \cong \frac{f''(k^*)c^*}{\theta}\frac{\tilde{k}}{\tilde{c}}$, and (14) $\frac{\dot{\tilde{k}}}{\tilde{k}} \cong \beta - \frac{\tilde{c}}{\tilde{k}}$.

Note that these are the same as equations (2.33) and (2.34) in the text; adding a positive depreciation rate does not alter the expressions for the growth rates of $\tilde{c}$ and $\tilde{k}$. Thus equation (2.38), the expression for μ, the constant growth rate of both $\tilde{c}$ and $\tilde{k}$ as the economy moves toward the balanced growth path, is still valid. Thus

(15) $\mu_1 = \frac{\beta - \sqrt{\beta^2 - 4f''(k^*)c^*/\theta}}{2}$,

where we have chosen the negative growth rate so that c and k are moving toward c^* and k^*, not away from them.

Now consider the Cobb-Douglas production function, $f(k) = k^\alpha$. Thus

(16) $f'(k^*) = \alpha k^{*\alpha-1} \equiv r^* + \delta$, and (17) $f''(k^*) = \alpha(\alpha - 1)k^{*\alpha-2}$.

Squaring both sides of equation (16) gives us

(18) $(r^* + \delta)^2 = \alpha^2 k^{*2\alpha-2}$,

and so equation (17) can be rewritten as

(19) $f''(k^*) = \frac{(r^* + \delta)^2(\alpha - 1)}{\alpha k^{*\alpha}} = \frac{\alpha - 1}{\alpha}\frac{(r^* + \delta)^2}{f(k^*)}$.

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In addition, defining s* to be the saving rate on the balanced growth path, we can write the balanced-growth-path level of consumption as

(20) $c^* = (1 - s^*)f(k^*)$.

Substituting equations (19) and (20) into (15) yields

$$(21)\quad \mu_1 = \frac{\beta - \sqrt{\beta^2 - 4\left(\frac{\alpha - 1}{\alpha}\right)\frac{(r^* + \delta)^2}{f(k^*)\theta}(1 - s^*)f(k^*)}}{2}.$$

Canceling the f(k*) and multiplying through by the minus sign yields

$$(22)\quad \mu_1 = \frac{\beta - \sqrt{\beta^2 + \frac{4}{\theta}\left(\frac{1 - \alpha}{\alpha}\right)(r^* + \delta)^2(1 - s^*)}}{2}.$$

On the balanced growth path, the condition required for $\dot{c} = 0$ is given by $r^* = \rho + \theta g$ and thus

(23) $r^* + \delta = \rho + \theta g + \delta$.

In addition, actual saving, s*f(k*), equals break-even investment, $(n + g + \delta)k^*$, and thus

$$(24)\quad s^* = \frac{(n + g + \delta)k^*}{f(k^*)} = \frac{(n + g + \delta)}{k^{*\alpha - 1}} = \frac{\alpha(n + g + \delta)}{(r^* + \delta)},$$

where we have used equation (16), $r^* + \delta = \alpha k^{*\alpha - 1}$.

From equation (24), we can write

$$(25)\quad (1 - s^*) = \frac{(r^* + \delta) - \alpha(n + g + \delta)}{(r^* + \delta)}.$$

Substituting equations (23) and (25) into equation (22) yields

$$(26)\quad \mu_1 = \frac{\beta - \sqrt{\beta^2 + \frac{4}{\theta}\left(\frac{1 - \alpha}{\alpha}\right)(\rho + \theta g + \delta)\left[\rho + \theta g + \delta - \alpha(n + g + \delta)\right]}}{2}.$$

Equation (26) is analogous to equation (2.40) in the text. It expresses the rate of adjustment in terms of the underlying parameters of the model. Keeping the values in the text – $\alpha = 1/3$, $\rho = 4\%$, $n = 2\%$, $g = 1\%$ and $\theta = 1$ – and using $\delta = 3\%$ yields a value for μ_1 of approximately - 8.8%. This is faster convergence than the -5.4% obtained with no depreciation.

Problem 2.9

(a) We are given that

(1) $y(t) = k(t)^{\alpha}$.

From the textbook we know that when $\dot{c} = 0$, then $f'(k) = \rho + \theta g$. Substituting (1) and simplifying results in

$$(2)\quad k^* = \left(\frac{\alpha}{\rho + \theta g}\right)^{\frac{1}{1 - \alpha}}.$$

(b) Likewise, from the textbook, we know that when $\dot{k} = 0$ then $c^* = f(k) - (n + g)k$. Substituting (1) and simplifying results in

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(3) $c^* = \left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{\alpha}{1-\alpha}} - (n+g)\left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{1}{1-\alpha}}$.

(c) Let $z(t) = k(t)/y(t)$ and $x(t) = c(t)/k(t)$. Substituting (1) into the first equation and simplifying, we get

(4) $k = z^{1/(1-\alpha)} \Leftrightarrow k^{1-\alpha} = z$.

Substituting (4) into the second equation above and simplifying results in

(5) $ck^{-\alpha} = xz$.

Now, using equation (4), take the time derivative of $z = k/y$, which results in

(6) $\dot{z} = (1-\alpha)k^{-\alpha}\dot{k}$.

Equation (2.26) in the textbook tells us that $\dot{k} = k^{\alpha} - c - (n+g)k$; therefore (6) becomes

(7) $\dot{z} = (1-\alpha)k^{-\alpha}(k^{\alpha} - c - (n+g)k)$.

Simplifying and substituting equations (4) and (5), equation (7) becomes

(8) $\dot{z} = (1-\alpha)(1 - xz - (n+g)z)$.

Now look at $x = c/k$. Taking logs and then the time derivative results in

(9) $\dot{x}/x = \dot{c}/c - \dot{k}/k$.

Using equation (2.25) and (2.26) from the textbook, equation (9) becomes

(10) $\frac{\dot{x}}{x} = \frac{\alpha k^{\alpha-1} - \rho - \theta g}{\theta} + \frac{-k^{\alpha} + c + (n+g)k}{k}$.

Again, substituting equations (4) and (5) into (10), and using the assumption that $\alpha = \theta$ gives us

(11) $\dot{x}/x = x + n - \rho/\alpha$.

(d)(i) From the conjecture that x is constant, equation (8) becomes

(12) $\dot{z} = (1-\alpha)(1 - (n+g+x^*)z)$.

To find the path of z, consider equation (12) and observe that it is a linear non-homogeneous ordinary differential equation. Therefore, our solution will consist of the complementary solution (z_c) and the particular solution (z_p). For simplification, let $\lambda = (1-\alpha)(n+g+x^*)$.

To solve for the complementary solution, we consider the homogeneous case in which $\dot{z} + \lambda z = 0$. Isolating for $\dot{z}/z$ yields a complementary solution of

(13) $z_c = A_1 e^{-\lambda t}$, where A_1 is a constant of integration.

To solve for the particular solution, we consider the non-homogeneous case, where $\dot{z} + \lambda z = (1-\alpha)$. Using an integrating factor, we find the solution to be

(14) $z_p = (1-\alpha)/\lambda + A_2 e^{-\lambda t}$, where A_2 is a constant of integration. Therefore,

(15) $z = z_p + z_c \Rightarrow z = (1-\alpha)/\lambda + (A_1 + A_2)e^{-\lambda t}$.

Using the initial condition z(0), we can substitute for $A_1 + A_2$ and (15) becomes

(16) $z = (1-\alpha)/\lambda + e^{-\lambda t}(z(0) - (1-\alpha)/\lambda)$.

Lastly, we can simplify the $(1-\alpha)/\lambda$ term by substituting equations (2) and (3) for x* and using equation (4), resulting in

(17) $z = z^* + e^{-\lambda t}(z(0) - z^*)$.

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(ii) We want to find the path of y, so consider equation (1) and substitute in equation (4). We know the path value of z, so substitute in equation (17). This results in

(18) $y = (z^* + e^{-\lambda t}(z(0) - z^*))^{\alpha/(1-\alpha)}$.

Again, use equation (4) to put everything in terms of k and (18) becomes

(19) $y = ((k^*)^{1-\alpha} + e^{-\lambda t}(k(0)^{1-\alpha} - (k^*)^{1-\alpha}))^{\alpha/(1-\alpha)}$.

Now we would like to see whether the speed of convergence to the balanced growth path is constant. Using equation (19) and subtracting the balanced growth path y*, where

$y^* = (k^*)^{\alpha} = \left(\frac{\alpha}{\rho + \alpha g}\right)^{\alpha/(1-\alpha)}$ from equation (2), and taking logs results in

(20) $\ln(y - y^*) = \ln\left(z^{\alpha/(1-\alpha)} - \left(\frac{\alpha}{\rho + \theta g}\right)^{\alpha/(1-\alpha)}\right)$.

Taking the time derivative, we get:

(21) $\frac{\partial \ln(y - y^*)}{\partial t} = \frac{\frac{\alpha}{1-\alpha} z^{\frac{\alpha}{1-\alpha}-1} \dot{z}}{z^{\frac{\alpha}{1-\alpha}} - \left(\frac{\alpha}{\rho + \theta g}\right)^{\frac{\alpha}{1-\alpha}}}$.

Clearly this is not constant, so the speed of convergence is not constant.

(e) We want to see whether equations (2.25) and (2.26) from the textbook— $\dot{c}/c = (\alpha k^{1-\alpha} - \rho - \theta g)/\theta$ and $\dot{k} = k^{\alpha} - c - (n + g)$ —are satisfied. Because these equations were used to find $\dot{x}/x$, then $\dot{x}/x$ holds if and only if equations (2.25) and (2.26) also hold. However, we know that equation (2.26) holds because it was used previously to find $\dot{z}$. Thus, we can say that $\dot{x}/x = 0$ if and only if $\dot{c}/c = 0$. Assuming that $\dot{x}/x = 0$ and using equation (11) results in

(22) $x^* = \rho/\alpha - n$.

Using parts (a) and (b), the balanced growth path of x* is $(\rho + \alpha g)/g - (n + g)$ so (22) becomes

(23) $x^* = \rho/\alpha - n$.

Equation (23) is the same result as (11), so equations (2.25) and (2.26) are satisfied.

Problem 2.10

(a) The real after-tax rate of return on capital is now given by (1 - τ)f ' (k(t)). Thus the household's maximization would now yield the following expression describing the dynamics of consumption per unit of effective labor:

(1) $\frac{\dot{c}(t)}{c(t)} = \frac{\left[(1-\tau)f'(k(t)) - \rho - \theta g\right]}{\theta}$.

The condition required for $\dot{c} = 0$ is given by (1 - τ)f ' (k) = ρ + θg. The after-tax rate of return must equal ρ + θg. Compared to the case without a tax on capital, f ' (k), the pre-tax rate of return on capital, must be higher and thus k must be lower for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left.

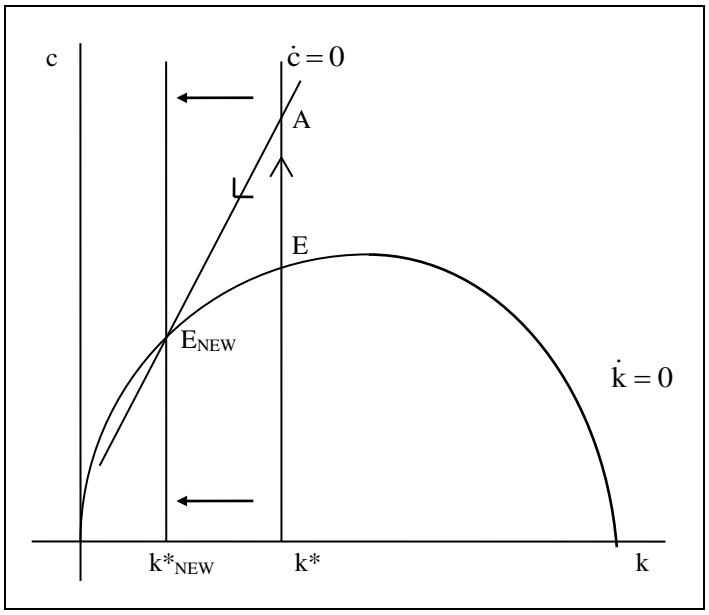
The equation describing the dynamics of the capital stock per unit of effective labor is still given by

(2) $\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$.

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For a given k, the level of c that implies $\dot{k} = 0$ is given by c(t) = f(k) - (n + g)k. Since the tax is rebated to households in the form of lump-sum transfers, this $\dot{k} = 0$ locus is unaffected.

(b) At time 0, when the tax is put in place, the value of k, the stock of capital per unit of effective labor, is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k* on the old balanced growth path.



In contrast, c, the rate at which households are consuming in units of effective labor, can jump at the time that the tax is introduced. This jump in c is not inconsistent with the consumption-smoothing behavior implied by the household's optimization problem since the tax was unexpected and could not be prepared for.

For the economy to reach the new balanced growth path, it should be clear what must occur. At time 0, c jumps up so that the economy is on the new saddle path. In the figure, the economy jumps from point E to a point such as A. Since the return to saving and accumulating capital is now lower than before, people switch away from saving and into consumption.

After time 0, the economy will gradually move down the new saddle path until it eventually reaches the new balanced growth path at E_{NEW} .

(c) On the new balanced growth path at E_{NEW} , the distortionary tax on investment income has caused the economy to have a lower level of capital per unit of effective labor as well as a lower level of consumption per unit of effective labor.

(d) (i) From the analysis above, we know that the higher is the tax rate on investment income, τ, the lower will be the balanced-growth-path level of k*, all else equal. In terms of the above story, the higher is τ the more that the $\dot{c} = 0$ locus shifts to the left and hence the more that k* falls. Thus $\partial k^*/\partial\tau < 0$.

On a balanced growth path, the fraction of output that is saved and invested, the saving rate, is given by [f(k*) - c*]/f(k*). Since k is constant, or $\dot{k} = 0$, on a balanced growth path then from $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$ we can write f(k*) - c* = (n + g)k*. Using this we can rewrite the fraction of output that is saved on a balanced growth path as

(3) s = [(n + g)k*]/f(k*).

Use equation (3) to take the derivative of the saving rate with respect to the tax rate, τ:

$$(4)\quad \frac{\partial s}{\partial \tau} = \frac{(n+g)(\partial k^*/\partial\tau) f(k^*) - (n+g)k^* f'(k^*)(\partial k^*/\partial\tau)}{f(k^*)^2}.$$

Simplifying yields

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(5) $\dfrac{\partial s}{\partial \tau} = \dfrac{(n+g)}{f(k^*)}\dfrac{\partial k^*}{\partial \tau} - \dfrac{(n+g)}{f(k^*)}\dfrac{k^* f'(k^*)}{f(k^*)}\dfrac{\partial k^*}{\partial \tau} = \dfrac{(n+g)}{f(k^*)}\dfrac{\partial k^*}{\partial \tau}\left[1 - \dfrac{k^* f'(k^*)}{f(k^*)}\right].$

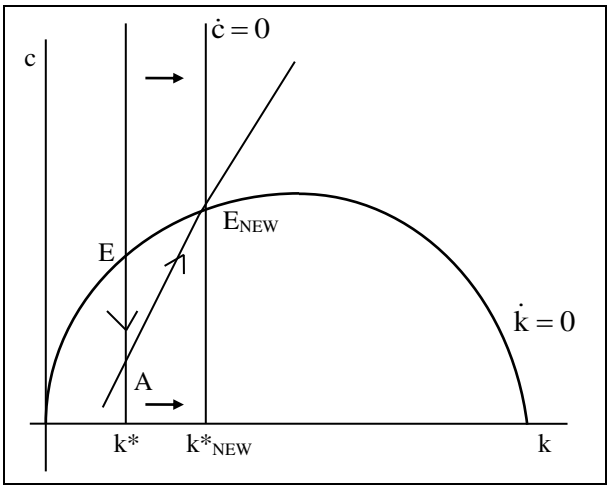
Recall that $k^*f'(k^*)/f(k^*) \equiv \alpha_K(k^*)$ is capital's (pre-tax) share in income, which must be less than one. Since $\partial k^*/\partial \tau < 0$ we can write

(6) $\dfrac{\partial s}{\partial \tau} = \dfrac{(n+g)}{f(k^*)}\dfrac{\partial k^*}{\partial \tau}\left[1 - \alpha_K(k^*)\right] < 0.$

Thus the saving rate on the balanced growth path is decreasing in τ.

(d) (ii) Citizens in low-τ, high-k^*, high-saving countries do not have the incentive to invest in low-saving countries. From part (a), the condition required for $\dot{c} = 0$ is $(1 - \tau)f'(k) = \rho + \theta g$. That is, the after-tax rate of return must equal $\rho + \theta g$. Assuming preferences and technology are the same across countries so that ρ, θ and g are the same across countries, the after-tax rate of return will be the same across countries. Since the after-tax rate of return is thus the same in low-saving countries as it is in high-saving countries, there is no incentive to shift saving from a high-saving to a low-saving country.

(e) Should the government subsidize investment instead and fund this with a lump-sum tax? This would lead to the opposite result from above and the economy would have higher c and k on the new balanced growth path.



The answer is no. The original market outcome is already the one that would be chosen by a central planner attempting to maximize the lifetime utility of a representative household subject to the capital-accumulation equation. It therefore gives the household the highest possible lifetime utility.

Starting at point E, the implementation of the subsidy would lead to a short-term drop in consumption at point A, but would eventually result in permanently higher consumption at point E_{NEW}.

It would turn out that the utility lost from the short-term sacrifice would outweigh the utility gained in the long-term (all in present value terms, appropriately discounted).

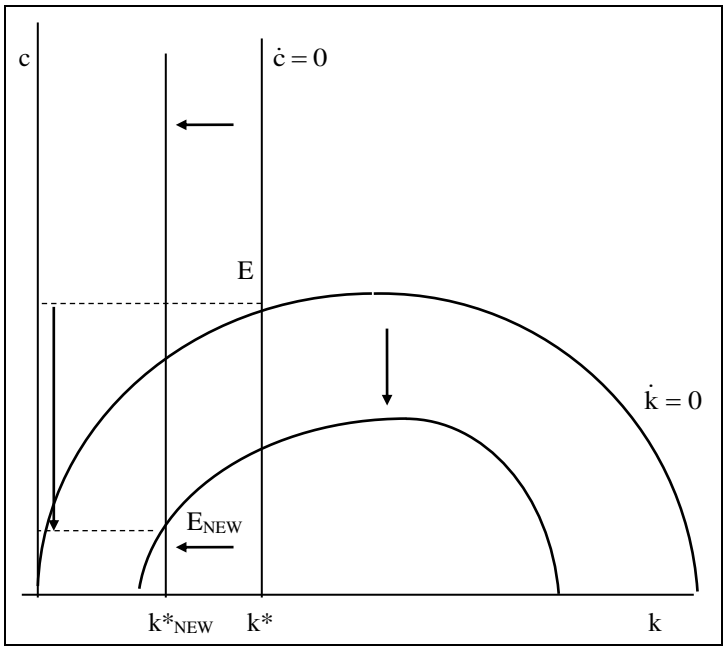
This is the same type of argument used to explain the reason that households do not choose to consume at the golden-rule level. See Section 2.4 for a more complete description of the welfare implications of this model.

(f) Suppose the government does not rebate the tax revenue to households but instead uses it to make government purchases. Let G(t) represent government purchases per unit of effective labor. The equation describing the dynamics of the capital stock per unit of effective labor is now given by

(2') $\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t).$

The fact that the government is making purchases that do not add to the capital stock – it is assumed to be government consumption, not government investment – shifts down the $\dot{k} = 0$ locus.

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After the imposition of the tax, the $\dot{c} = 0$ locus shifts to the left, just as it did in the case in which the government rebated the tax to households. In the end, k* falls to k^*_{NEW} just as in the case where the government rebated the tax.

Consumption per unit of effective labor on the new balanced growth path at E_{NEW} is lower than in the case where the tax is rebated by the amount of the government purchases, which is $\tau f'(k)k$.

Finally, whether the level of c jumps up or down initially depends upon whether the new saddle path passes above or below the original balanced-growth-path point of E.

Problem 2.11

(a) - (c) Before the tax is put in place, i.e. until time t_1, the equations governing the dynamics of the economy are

(1) $$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta},$$

and

(2) $$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$$

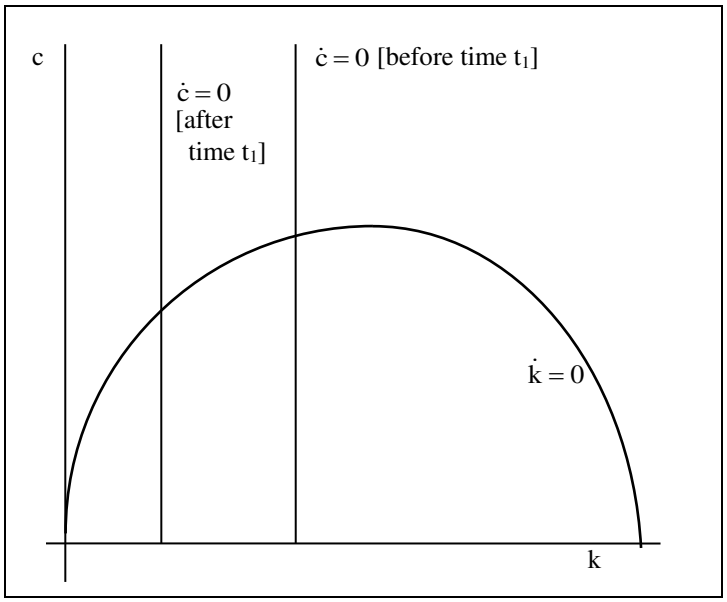
The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. The capital-accumulation equation is not affected when the tax is put in place at time t_1 since we are assuming that the government is rebating the tax, not spending it.

Since the real after-tax rate of return on capital is now $(1 - \tau)f'(k(t))$, the household's maximization yields the following growth rate of consumption:

(3) $$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau)f'(k(t)) - \rho - \theta g}{\theta}.$$

The condition required for $\dot{c} = 0$ is now given by $(1 - \tau)f'(k) = \rho + \theta g$. The after-tax rate of return on capital must equal $\rho + \theta g$. Thus the pre-tax rate of return, $f'(k)$, must be higher and thus k must be lower in order for $\dot{c} = 0$. Thus at time t_1, the $\dot{c} = 0$ locus shifts to the left.

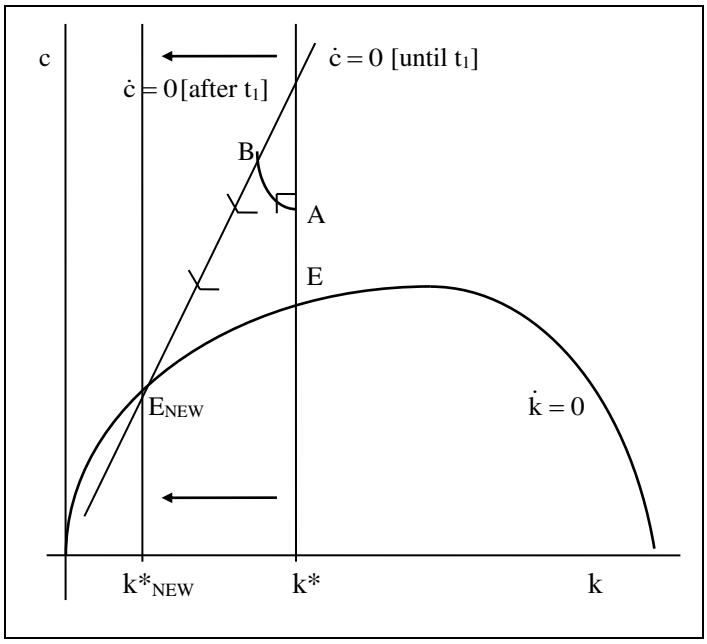
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The important point is that the dynamics of the economy are still governed by the original equations of motion until the tax is put in place. Between the time of the announcement and the time the tax is put in place, it is the original $\dot{c} = 0$ locus that is relevant.

When the tax is put in place at time t_1, c cannot jump discontinuously because households know ahead of time that the tax will be implemented then. A discontinuous jump in c would be inconsistent with the consumption smoothing implied by the household's intertemporal optimization. The household would not want c to be low, and thus marginal utility to be high, a moment before t_1 knowing that c will jump up and be high, and thus marginal utility will be low, a moment after t_1. The household would like to smooth consumption between the two instants in time.

(d) We know that c cannot jump at time t_1. We also know that if the economy is to reach the new balanced growth path at point E_{NEW}, it must be right on the new saddle path at the time that the tax is put in place. Thus when the tax is announced at time t_0, c must jump up to a point such as A. Point A lies between the original balanced growth path at E and the new saddle path.

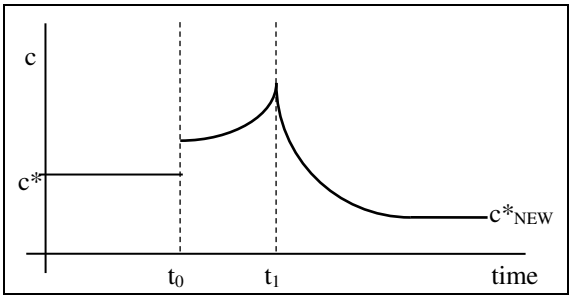


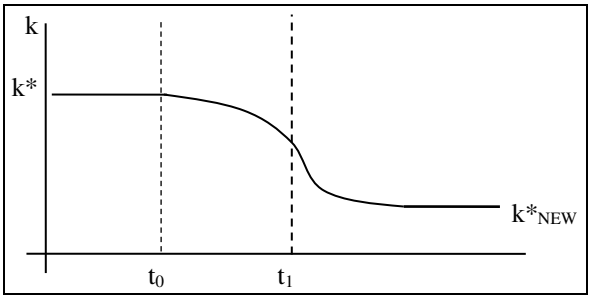
At A, c is too high to maintain the capital stock at k* and so k begins falling. Between t_0 and t_1, the dynamics of the system are still governed by the original $\dot{c} = 0$ locus. The economy is thus to the left of the $\dot{c} = 0$ locus and so consumption begins rising.

The economy moves off to the northwest until at t_1, it is right at point B on the new saddle path. The tax is then put in place and the system is governed by the new $\dot{c} = 0$ locus. Thus c begins falling. The economy moves down the new saddle path, eventually reaching point E_{NEW}.

(e) The story in part (d) implies the following time paths for consumption per unit of effective labor and capital per unit of effective labor.

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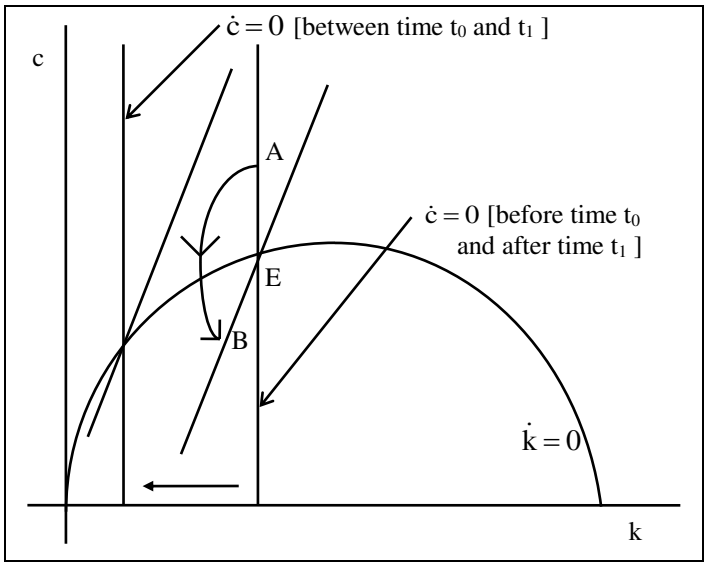
Problem 2.12

(a) The first point is that consumption cannot jump at time t_1. Households know ahead of time that the tax will end then and so a discontinuous jump in c would be inconsistent with the consumption-smoothing behavior implied by the household's intertemporal optimization. Thus, for the economy to return to a balanced growth path, we must be somewhere on the original saddle path right at time t_1.

Before the tax is put in place – until time t_0 – and after the tax is removed – after time t_1 – the equations governing the dynamics of the economy are

$$(1)\quad \frac{\dot{c}(t)}{c(t)} = \frac{f'\big(k(t)\big) - \rho - \theta g}{\theta}, \qquad \text{and} \qquad (2)\quad \dot{k}(t) = f\big(k(t)\big) - c(t) - (n+g)k(t).$$

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. The capital-accumulation equation, and thus the $\dot{k} = 0$ locus, is not affected by the tax. The $\dot{c} = 0$ locus is affected, however. Between time t_0 and time t_1, the condition required for $\dot{c} = 0$ is that the after-tax rate of return on capital equal $\rho + \theta g$ so that $(1 - \tau)f'(k) = \rho + \theta g$. Thus between t_0 and t_1, $f'(k)$ must be higher and so k must be lower in order for $\dot{c} = 0$. That is, between time t_0 and time t_1, the $\dot{c} = 0$ locus lies to the left of its original position.



At time t_0, the tax is put in place. At point E, the economy is still on the $\dot{k} = 0$ locus but is now to the right of the new $\dot{c} = 0$ locus. Thus if c did not jump up to a point like A, c would begin falling. The economy would then be below the $\dot{k} = 0$ locus and so k would start rising. The economy would drift away from point E in the direction of the southeast and could not be on the original saddle path right at time t_1.

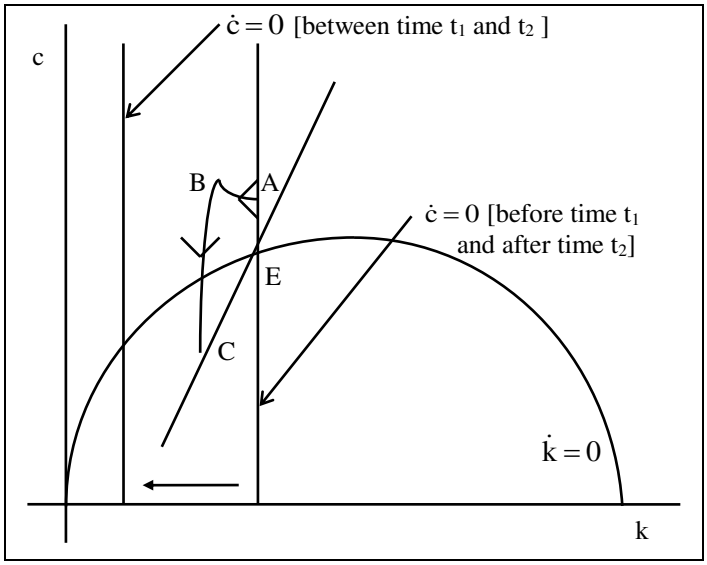
Thus at time t_0, c must jump up so that the economy is at a point like A. Thus, k and c begin falling. Eventually the economy crosses the $\dot{k} = 0$ locus and so k begins rising. This can be interpreted as households anticipating the removal of the tax on capital and thus being willing to accumulate capital again. Point A must be such that given the dynamics of the system, the economy is right at a point like B, on the original saddle path, at time t_1 when the tax is removed. After t_1, the original $\dot{c} = 0$ locus governs the dynamics of the system again. Thus the economy moves up the original saddle path, eventually returning to the original balanced growth path at point E.

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(b) The first point is that consumption cannot jump at either time t_1 or time t_2. Households know ahead of time that the tax will be implemented at t_1 and removed at t_2. Thus a discontinuous jump in c at either date would be inconsistent with the consumption-smoothing behavior implied by the household's intertemporal optimization.

In order for the economy to return to a balanced growth path, the economy must be somewhere on the original saddle path right at time t_2.

Before the tax is put in place – until time t_1 – and after the tax is removed – after time t_2 – equations (1) and (2) govern the dynamics of the system. An important point is that even during the time between the announcement and the implementation of the tax – that is, between time t_0 and time t_1 – the original $\dot{c} = 0$ locus governs the dynamics of the system.



At time t_0, the tax is announced. Consumption must jump up so that the economy is at a point like A.

At A, the economy is still on the $\dot{c} = 0$ locus but is above the $\dot{k} = 0$ locus and so k starts falling. The economy is then to the left of the $\dot{c} = 0$ locus and so c starts rising. The economy drifts off to the northwest.

At time t_1, the tax is implemented, the $\dot{c} = 0$ locus shifts to the left and the economy is at a point like B. The economy is still above the $\dot{k} = 0$ locus but is now to the right of the relevant $\dot{c} = 0$ locus; k continues to fall and c stops rising and begins to fall.

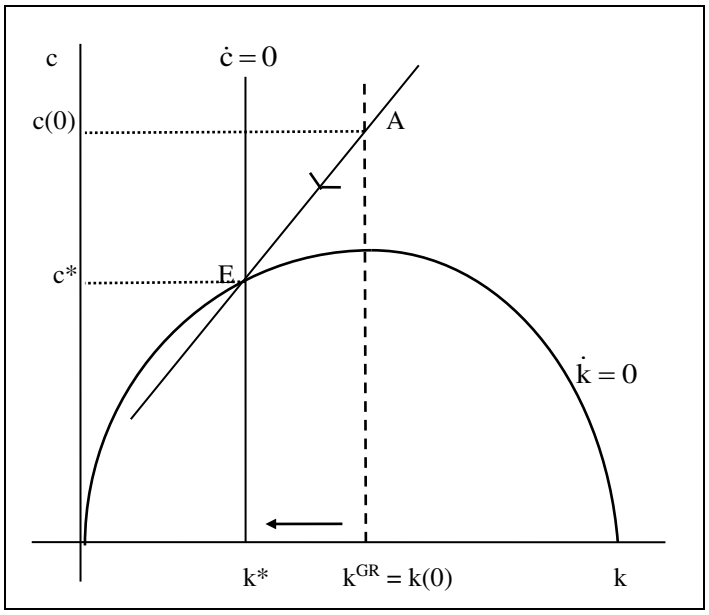
Eventually the economy crosses the $\dot{k} = 0$ locus and k begins rising. Households begin accumulating capital again before the actual removal of the tax on capital income. Point A must be chosen so that given the dynamics of the system, the economy is right at a point like C, on the original saddle path, at time t_2 when the tax is removed.

After t_2, the original $\dot{c} = 0$ locus governs the dynamics of the system again. Thus the economy moves up the original saddle path, eventually returning to the original balanced growth path at point E.

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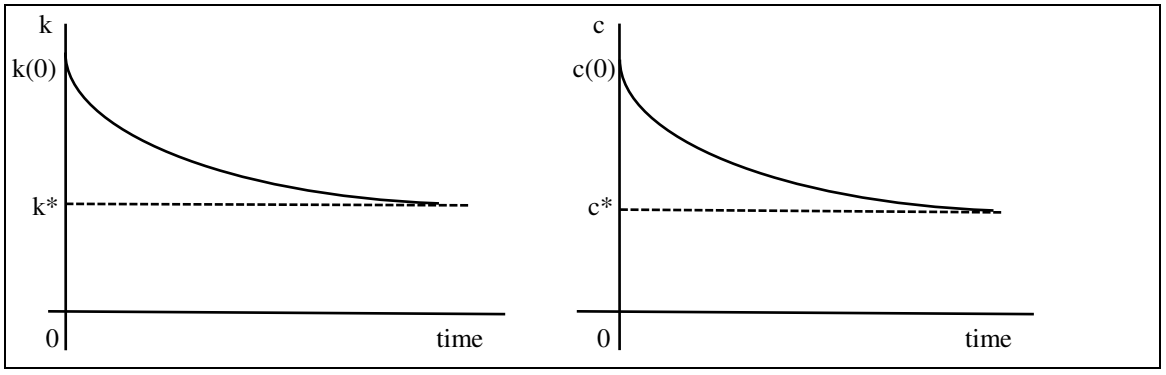
Problem 2.13

(a) This case represents normal saddle-path convergence. The economy would start with $k(0) = k^{GR}$ and consumption equal to c(0), the value that would put the economy on the saddle path. See point A in the figure at right.



At point A, we are above the $\dot{k} = 0$ locus and so k would fall over time. Since we are to the right of the $\dot{c} = 0$ locus, consumption would fall over time. The economy would move down along the saddle path with k and c asymptotically approaching c* and k* at point E.

See the figures below for the time paths of k and c.



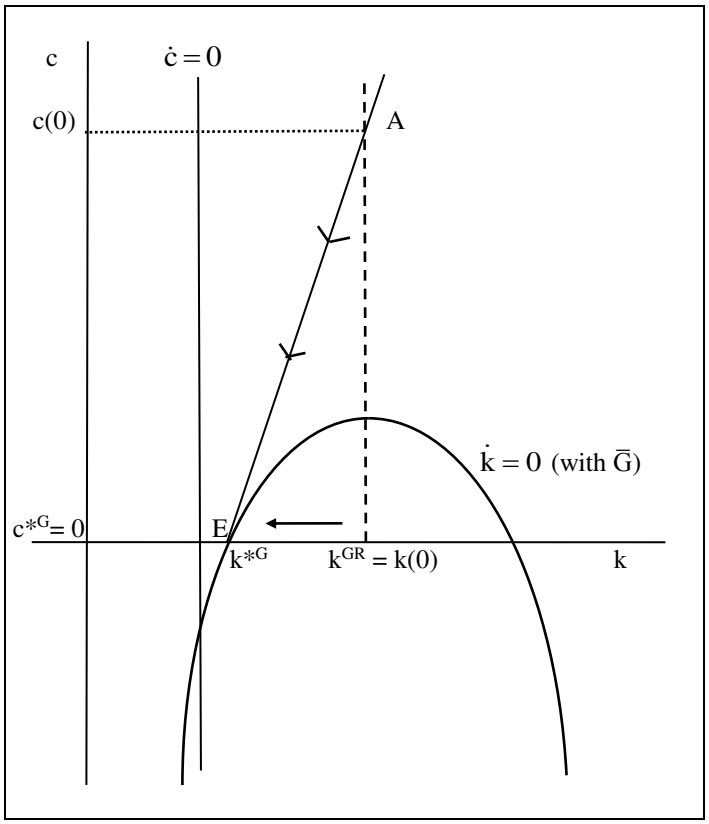
(b) In the version of the model with government, the equation of motion for k is given by equation (2.41) in the text, which is

(1) $\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n + g)k(t)$.

Thus, for a given value of government purchases, $\bar{G}$, the level of c that implies $\dot{k} = 0$ is given by $f(k) - (n + g)k - \bar{G}$.

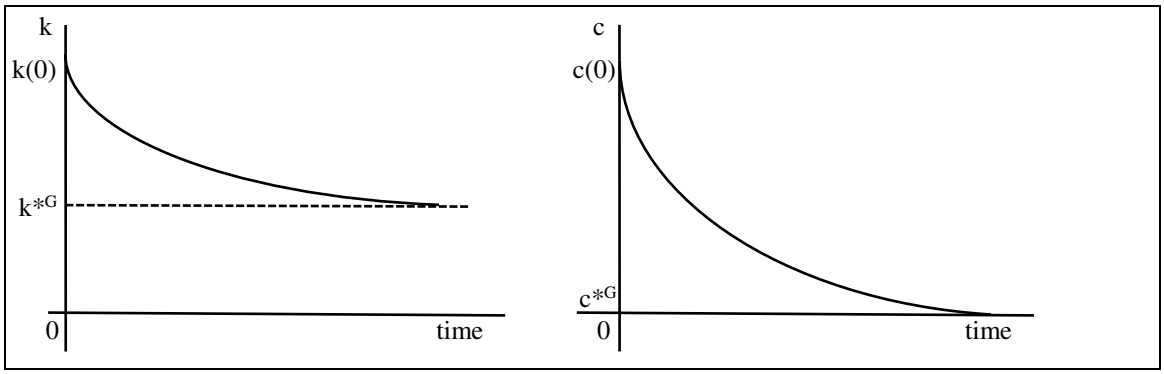
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To compare the $\dot{k} = 0$ locus here to the situation in the solution to part (a) in which government purchases are effectively 0, note that the locus will shift down by the amount of the constant level of government purchases, $\bar{G}$. Since we are told that $\bar{G}$ is strictly less than $f(k^{GR}) - (n + g)k^{GR}$, we know that the new $\dot{k} = 0$ locus lies above the x-axis at k^{GR} , i.e. that the level of c consistent with $\dot{k} = 0$ at k^{GR} is positive. We are also told that $\bar{G}$ is strictly greater than $f(k^*) - (n + g)k^*$, where k^* is the balanced-growth-path value from part (a) where G was equal to 0. This means that the new $\dot{k} = 0$ locus lies below the x-axis at k^*, i.e. that it implies a negative value of c at the level of k^* from part (a). See the figure at right. Thus the balanced growth path from part (a) is no longer feasible since it would imply negative consumption at some point.



The balanced growth path in this case is at the point where the $\dot{k} = 0$ locus crosses the x-axis, labeled E in the figure at right. Thus starting from point A, consumption and the capital stock gradually decrease until we approach the point at which $f(k) - (n + g)k = \bar{G}$ and $c = 0$. How do we know that this solution dominates the intuitively appealing situation in which consumption simply drops to the level consistent with $\dot{k} = 0$ at k^{GR}, and then stays there? Staying at that point would imply that the household would not be behaving optimally with respect to its consumption and saving decision. By consuming a little bit more, the household would be able to increase total utility because the cost of borrowing to do so would be less than the household's discount rate. The same holds for all points along the $\dot{k} = 0$ locus as consumption approaches 0. Other points along the x-axis cannot be equilibria because capital would not be constant. If k were less than k^{*G}, capital would eventually become negative, which is not possible.

The figures below show the time paths of k and c in this case.



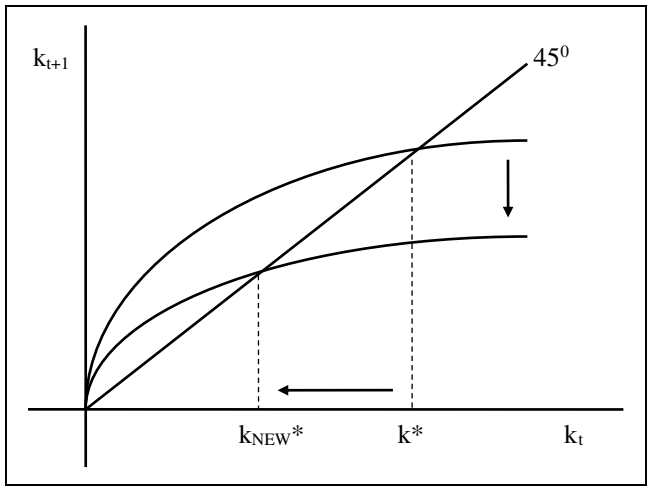
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Problem 2.14

Equation (2.61) in the text describes the relationship between k_{t+1} and k_t in the special case of logarithmic utility and Cobb-Douglas production:

(2.61) $k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}(1-\alpha)k_t^{\alpha}$.

(a) A rise in n shifts the k_{t+1} function down. From equation (2.61), a higher n means a smaller k_{t+1} for a given k_t. Since the fraction of their labor income that the young save does not depend on n, a given amount of capital per unit of effective labor and thus output per unit of effective labor in time t yields the same amount of saving in period t. Thus it yields the same amount of capital in period t + 1. However, the number of individuals increases more from period t to period t + 1 than it used to. So that capital is spread out among more individuals than it would have been in the absence of the increase in population growth and thus capital per unit of effective labor in period t + 1 is lower for a given k_t.



(b) With the parameter B added to the Cobb-Douglas production function, $f(k) = Bk^{\alpha}$, equation (2.61) becomes

(1) $k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}(1-\alpha)Bk_t^{\alpha}$.

This fall in B causes the k_{t+1} function to shift down. See the figure from part (a). A lower B means that a given amount of capital per unit of effective labor in period t now produces less output per unit of effective labor in period t. Since the fraction of their labor income that the young save does not depend on B, this leads to less total saving and a lower capital stock per unit of effective labor in period t + 1 for a given k_t.

(c) We need to determine the effect on k_{t+1} for a given k_t, of a change in α. From equation (2.61):

(2) $\frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}\left[-k_t^{\alpha} + (1-\alpha)\frac{\partial k_t^{\alpha}}{\partial \alpha}\right]$.

We need to determine $\partial k_t^{\alpha}/\partial \alpha$. Define $f(\alpha) \equiv k_t^{\alpha}$ and note that $\ln f(\alpha) = \alpha \ln k_t$. Thus

(3) $\partial \ln f(\alpha)/\partial \alpha = \ln k_t$.

Now note that we can write

(4) $\frac{\partial f(\alpha)}{\partial \alpha} = \frac{\partial f(\alpha)}{\partial \ln f(\alpha)} \frac{\partial \ln f(\alpha)}{\partial \alpha} = \frac{1}{\left[\partial \ln f(\alpha)/\partial f(\alpha)\right]} \frac{\partial \ln f(\alpha)}{\partial \alpha}$,

and thus finally

(5) $\partial f(\alpha)/\partial \alpha = f(\alpha) \ln k_t$.

Therefore, we have $\partial k_t^{\alpha}/\partial \alpha = k_t^{\alpha} \ln k_t$. Substituting this fact into equation (2) yields

(6) $\frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}\left[-k_t^{\alpha} + (1-\alpha)k_t^{\alpha} \ln k_t\right]$,

or simply

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(7) $\dfrac{\partial k_{t+1}}{\partial \alpha} = \dfrac{1}{(1+n)(1+g)} \dfrac{1}{2+\rho} \left\{ k_t^{\alpha} \left[(1-\alpha) \ln k_t - 1 \right] \right\}$.

Thus, for $(1 - \alpha)\ln k_t - 1 > 0$, or $\ln k_t > 1/(1 - \alpha)$, an increase in α means a higher k_{t+1} for a given k_t and thus the k_{t+1} function shifts up over this range of k_t 's. However, for $\ln k_t < 1/(1 - \alpha)$, an increase in α means a lower k_{t+1} for a given k_t . Thus the k_{t+1} function shifts down over this range of k_t 's. Finally, right at $\ln k_t = 1/(1 - \alpha)$, the old and new k_{t+1} functions intersect.

Problem 2.15

(a) We need to find an expression for k_{t+1} as a function of k_t. Next period's capital stock is equal to this period's capital stock, plus any investment done this period, less any depreciation that occurs. Thus

(1) $K_{t+1} = K_t + sY_t - \delta K_t$.

To convert this into units of effective labor, divide both sides of equation (1) by $A_{t+1}L_{t+1}$:

(2) $\dfrac{K_{t+1}}{A_{t+1}L_{t+1}} = \dfrac{K_t(1-\delta) + sY_t}{A_{t+1}L_{t+1}} = \dfrac{K_t(1-\delta) + sY_t}{(1+n)(1+g)A_tL_t} = \dfrac{k_t(1-\delta) + sf(k_t)}{(1+n)(1+g)}$,

which simplifies to

(3) $k_{t+1} = \left[\dfrac{1-\delta}{(1+n)(1+g)} \right] k_t + \left[\dfrac{s}{(1+n)(1+g)} \right] f(k_t)$.

(b) We need to sketch k_{t+1} as a function of k_t .

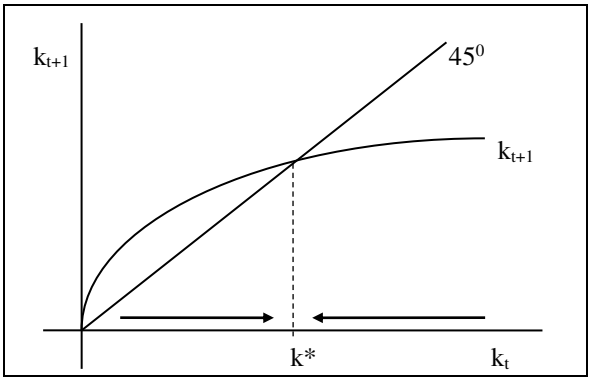
Note that

(4) $\dfrac{\partial k_{t+1}}{\partial k_t} = \dfrac{1-\delta}{(1+n)(1+g)} + \left[\dfrac{s}{(1+n)(1+g)} \right] f'(k_t) > 0$, and (5) $\dfrac{\partial^2 k_{t+1}}{\partial k_t^{\,2}} = \dfrac{sf''(k_t)}{(1+n)(1+g)} < 0$.

The Inada conditions are given by

(6) $\lim\limits_{k \to 0} \dfrac{\partial k_{t+1}}{\partial k_t} = \infty$, and (7) $\lim\limits_{k \to \infty} \dfrac{\partial k_{t+1}}{\partial k_t} = \dfrac{1-\delta}{(1+n)(1+g)} < 1$.

Thus the function eventually has a slope of less than one and will therefore cross the 45 degree line at some point. Also, the function is well-behaved and will cross the 45 degree line only once.



As long as k starts out at some value other than 0, the economy will converge to k*. For example, if k starts out below k*, we see that k_{t+1} will be greater than k_t and the economy will move toward k*. Similarly, if k starts out above k*, we see that k_{t+1} will be below k_t and again the economy will move toward k*. At k*, y* = f(k*) is also constant and we have a balanced growth path.

(c) On a balanced growth path, $k_{t+1} = k_t \equiv k^*$ and thus from equation (3) we have

(8) $k^* = \left[\dfrac{1-\delta}{(1+n)(1+g)} \right] k^* + \left[\dfrac{s}{(1+n)(1+g)} \right] f(k^*)$,

which simplifies to

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(9) $k^*\left[\frac{1+n+g+ng-1+\delta}{(1+n)(1+g)}\right]=\left[\frac{s}{(1+n)(1+g)}\right]f(k^*)$.

Thus on a balanced growth path:

(10) $k^*(n + g + ng + \delta) = sf(k^*)$.

Rearranging equation (10) to get an expression for s on the balanced growth path yields

(11) $s = (n + g + ng + \delta)k^*/f(k^*)$.

Consumption per unit of effective labor on the balanced growth path is given by

(12) $c^* = (1 - s)f(k^*)$.

Substitute equation (11) into equation (12):

(13) $c^*=\left[1-\frac{(n+g+ng+\delta)k^*}{f(k^*)}\right]f(k^*)=\left[\frac{f(k^*)-k^*(n+g+ng+\delta)}{f(k^*)}\right]f(k^*)$.

Canceling the $f(k^*)$ yields

(14) $c^* = f(k^*) - (n + g + ng + \delta)k^*$.

To get an expression for the $f'(k^*)$ that maximizes consumption per unit of effective labor on the balanced growth path, we need to maximize c^* with respect to k^*. The first-order condition is given by

(15) $\partial c^*/\partial k^* = f'(k^*) - (n + g + ng + \delta) = 0$.

Thus the golden-rule capital stock is defined implicitly by

(16) $f'(k_{GR}) = (n + g + ng + \delta)$.

(d) (i) Substitute a Cobb-Douglas production function, $f(k_t) = k_t^{\alpha}$, into equation (3):

(17) $k_{t+1}=\left[\frac{1-\delta}{(1+n)(1+g)}\right]k_t+\left[\frac{s}{(1+n)(1+g)}\right]k_t^{\alpha}$.

(d) (ii) On a balanced growth path, $k_{t+1} = k_t \equiv k^*$. Thus from equation (17):

(18) $k^*=\left[\frac{1-\delta}{(1+n)(1+g)}\right]k^*+\left[\frac{s}{(1+n)(1+g)}\right]k^{*\alpha}$.

Simplifying yields

(19) $\left[\frac{(1+n)(1+g)-(1-\delta)}{(1+n)(1+g)}\right]k^*=\left[\frac{s}{(1+n)(1+g)}\right]k^{*\alpha}$,

or

(20) $k^{*1-\alpha} = s/(n + g + ng + \delta)$.

Thus, finally we have

(21) $k^* = [s/(n + g + ng + \delta)]^{1/(1-\alpha)}$.

(d) (iii) Using equation (17):

(22) $\left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*}=\frac{1-\delta}{(1+n)(1+g)}+\frac{\alpha s}{(1+n)(1+g)}k^{*\alpha-1}$.

Substituting the balanced-growth-path value of k^* – equation (21) – into equation (22) yields

(23) $\left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*}=\frac{1-\delta}{(1+n)(1+g)}+\frac{\alpha s}{(1+n)(1+g)}\left[\frac{(n+g+ng+\delta)}{s}\right]$.

It will be useful to write $(n + g + ng + \delta)$ as $(1 + n)(1 + g) - (1 - \delta)$:

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(24) $$\left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} = \frac{(1-\delta)+\alpha\left[(1+n)(1+g)-(1-\delta)\right]}{(1+n)(1+g)}.$$

Simplifying further yields

(25) $$\left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} = \alpha + \frac{(1-\delta)(1-\alpha)}{(1+n)(1+g)}.$$

Replacing equation (17) by its first-order Taylor approximation around k = k* therefore gives us

(26) $k_{t+1} \cong k^* + \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]\left[k_t - k^*\right]$.

Since we can write this simply as

(27) $k_{t+1} - k^* \cong \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]\left[k_t - k^*\right]$,

equation (26) implies

(28) $k_t - k^* \cong \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]^t\left[k_0 - k^*\right]$.

Thus the economy moves fraction $1-\left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g)\right]$ of the way to the balanced growth path each period. Some simple algebra simplifies the expression for this rate of convergence to $(1-\alpha)(n+g+ng+\delta)/(1+n)(1+g)$. With α = 1/3, n = 1%, g = 2% and δ = 3%, this yields a rate of convergence of about 3.9%. This is slower than the rate of convergence found in the continuous-time Solow model.

Problem 2.16

(a) The individual's optimization problem is not affected by the depreciation which means that $r_t = f'(k_t) - \delta$. The household's problem is still to maximize utility as given by

(1) $$U_t = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta},$$

subject to the budget constraint

(2) $$C_{1,t} + \frac{1}{1+r_{t+1}}C_{2,t+1} = A_t w_t.$$

As in the text, with no depreciation, the fraction of income saved, $s(r_{t+1}) \equiv (1-C_{1,t})A_t w_t$, is given by

(3) $$s(r_{t+1}) = \frac{1}{1+(1+\rho)^{1/\theta}(1+r_{t+1})^{(\theta-1)/\theta}}.$$

Thus the way in which the fraction of income saved depends on the real interest rate, r_{t+1}, is unchanged. The only difference is that the real interest rate itself is now $f'(k_{t+1}) - \delta$, rather than just $f'(k_{t+1})$. The capital stock in period t + 1 equals the amount saved by young individuals in period t. Thus

(4) $K_{t+1} = S_t L_t$,

where S_t is the amount of saving done by a young person in period t. Note that $S_t \equiv s(r_{t+1})A_t w_t$; the amount of saving done is equal to the fraction of income saved times the amount of income. Thus equation (4) can be rewritten as

(5) $K_{t+1} = L_t s(r_{t+1})A_t w_t$.

To get this into units of time t + 1 effective labor, divide both sides of equation (5) by $A_{t+1}L_{t+1}$:

(6) $$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{A_t L_t}{A_{t+1}L_{t+1}}\left[s(r_{t+1})w_t\right].$$

Since $A_{t+1} = (1+g)A_t$, we have $A_t/A_{t+1} = 1/(1+g)$. Similarly, $L_t/L_{t+1} = 1/(1+n)$. In addition, $K_{t+1}/A_{t+1}L_{t+1} \equiv k_{t+1}$. Thus

(7) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[s(r_{t+1})w_t\right].$$

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Finally, substitute for $r_{t+1} = f'(k_{t+1}) - \delta$ and $w_t = f(k_t) - k_t f'(k_t)$:

(8) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[s\left(f'(k_{t+1}) - \delta\right)\right]\left[f(k_t) - k_t f'(k_t)\right].$$

This should be compared with equation (2.60) in the text, the analogous expression with no depreciation, which is

(2.60) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[s\left(f'(k_{t+1})\right)\right]\left[f(k_t) - k_t f'(k_t)\right].$$

Thus adding depreciation does alter the relationship between k_{t+1} and k_t. Whether k_{t+1} will be higher or lower for a given k_t depends on the way in which saving varies with r_{t+1}.

(b) With logarithmic utility, the fraction of income saved does not depend upon the rate of return on saving and in fact

(9) $s(r_{t+1}) = 1/(2 + \rho)$.

In addition, with Cobb-Douglas production, $y_t = k_t^{\alpha}$, the real wage is $w_t = k_t^{\alpha} - k_t \alpha k_t^{\alpha-1} = (1 - \alpha)k_t^{\alpha}$. Thus equation (8) becomes

(10) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[\frac{1}{2+\rho}(1-\alpha)k_t^{\alpha}\right].$$

We need to compare this with equation (3) in the solution to Problem 2.15, the analogous expression in the discrete-time Solow model, with the additional assumption of 100% depreciation (i.e. $\delta = 1$).

The saving rate in this economy is total saving divided by total output. Note that this is not the same as $s(r_{t+1})$, which is simply the fraction of their labor income that the young save. Denote the economy's total saving rate as $\hat{s}$. Then $\hat{s}$ will equal the saving of the young plus the dissaving of the old, all divided by total output and in addition, all variables are measured in units of effective labor.

The saving of the young is $\left[1/(2+\rho)\right](1-\alpha)k_t^{\alpha}$. Since there is 100% depreciation, the old do not get to dissave by the amount of the capital stock; there is no dissaving by the old. Thus

(11) $$\hat{s} = \frac{\left[1/(2+\rho)\right](1-\alpha)k_t^{\alpha}}{k_t^{\alpha}} = \frac{1}{2+\rho}(1-\alpha).$$

Thus equation (10) can be rewritten as

(12) $$k_{t+1} = \frac{1}{(1+n)(1+g)}\hat{s}k_t^{\alpha} = \left[\frac{\hat{s}}{(1+n)(1+g)}\right]f(k_t).$$

Note that this is the same as the expression for k_{t+1} as a function of k_t in the discrete-time Solow model with $\delta = 1$. That is, it is equivalent to equation (3) in the solution to Problem 2.15 with δ set to one. Thus that version of the Solow model does have some microeconomic foundations, although the assumption of 100% depreciation is quite unrealistic.

Problem 2.17

(a) (i) The utility function is given by

(1) $$\ln C_{1,t} + \left[1/(1+\rho)\right]\ln C_{2,t+1}.$$

With the social security tax of T per person, the individual faces the following constraints (with g, the growth rate of technology, equal to 0, A is simply a constant throughout):

(2) $C_{1,t} + S_t = Aw_t - T$, and

(3) $C_{2,t+1} = (1 + r_{t+1})S_t + (1+n)T$,

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where S_t represents the individual's saving in the first period. As far as the individual is concerned, the rate of return on social security is (1 + n); in general this will not be equal to the return on private saving which is $(1 + r_{t+1})$. From equation (3), $(1+r_{t+1})S_t = C_{2,t+1} - (1+n)T$. Solving for S_t yields

(4) $S_t = \dfrac{C_{2,t+1}}{1+r_{t+1}} - \dfrac{(1+n)}{(1+r_{t+1})}T.$

Now substitute equation (4) into equation (2):

(5) $C_{1,t} + \dfrac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - T + \dfrac{(1+n)}{(1+r_{t+1})}T.$

Rearranging, we get the intertemporal budget constraint:

(6) $C_{1,t} + \dfrac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - \dfrac{(r_{t+1}-n)}{(1+r_{t+1})}T.$

We know that with logarithmic utility, the individual will consume fraction $(1 + \rho)/(2 + \rho)$ of her lifetime wealth in the first period. Thus

(7) $C_{1,t} = \left(\dfrac{1+\rho}{2+\rho}\right)\left[Aw_t - \left(\dfrac{r_{t+1}-n}{1+r_{t+1}}\right)T\right].$

To solve for saving per person, substitute equation (7) into equation (2):

(8) $S_t = Aw_t - \left(\dfrac{1+\rho}{2+\rho}\right)\left[Aw_t - \left(\dfrac{r_{t+1}-n}{1+r_{t+1}}\right)T\right] - T$

Simplifying gives us

(9) $S_t = \left[1 - \left(\dfrac{1+\rho}{2+\rho}\right)\right]Aw_t - \left[1 - \left(\dfrac{1+\rho}{2+\rho}\right)\left(\dfrac{r_{t+1}-n}{1+r_{t+1}}\right)\right]T,$

or

(10) $S_t = \left[1/(2+\rho)\right]Aw_t - \left[\dfrac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}\right]T.$

Note that if $r_{t+1} = n$, saving is reduced one-for-one by the social security tax. If $r_{t+1} > n$, saving falls less than one-for-one. Finally, if $r_{t+1} < n$, saving falls more than one-for-one.

Denote $Z_t \equiv \left[(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)\right]/(2+\rho)(1+r_{t+1})$ and thus equation (10) becomes

(11) $S_t = \left[1/(2+\rho)\right]Aw_t - Z_t T.$

It is still true that the capital stock in period t+1 will be equal to the total saving of the young in period t, hence

(12) $K_{t+1} = S_t L_t$.

Converting this into units of effective labor by dividing both sides of (12) by AL_{t+1} and using equation (11) yields

(13) $k_{t+1} = \left[1/(1+n)\right]\left[\left(1/(2+\rho)\right)w_t - Z_t T/A\right].$

With a Cobb-Douglas production function, the real wage is given by

(14) $w_t = (1 - \alpha)k_t^{\alpha}$.

Substituting (14) into (13) gives the new relationship between capital in period t + 1 and capital in period t, all in units of effective labor:

(15) $k_{t+1} = \left[1/(1+n)\right]\left[\left(1/(2+\rho)\right)(1-\alpha)k_t^{\alpha} - Z_t T/A\right].$

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(a) (ii) To see what effect the introduction of the social security system has on the balanced-growth-path value of k, we must determine the sign of Z_t. If it is positive, the introduction of the tax, T, shifts down the k_{t+1} curve and reduces the balanced-growth-path value of k. We have

(16) $$Z_t = \frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})} = \frac{(1+1+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})},$$

and simplifying further allows us to determine the sign of Z_t :

(17) $$Z_t = \frac{(1+r_{t+1}) + (1+\rho)\left[(1+r_{t+1}) - (r_{t+1}-n)\right]}{(2+\rho)(1+r_{t+1})} = \frac{(1+r_{t+1}) + (1+\rho)(1+n)}{(2+\rho)(1+r_{t+1})} > 0.$$

Thus, the k_{t+1} curve shifts down, relative to the case without the social security, and k* is reduced.

(a) (iii) If the economy was initially dynamically efficient, a marginal increase in T would result in a gain to the old generation that would receive the extra benefits. However, it would reduce k* further below k_{GR} and thus leave future generations worse off, with lower consumption possibilities. If the economy was initially dynamically inefficient, so that $k^* > k_{GR}$, the old generation would again gain due to the extra benefits. In this case, the reduction in k* would allow for higher consumption for future generations and would be welfare-improving. The introduction of the tax in this case would reduce or possibly eliminate the dynamic inefficiency caused by the over-accumulation of capital.

(b) (i) Equation (3) becomes

(18) $$C_{2,t+1} = (1+r_{t+1})S_t + (1+r_{t+1})T.$$

As far as the individual is concerned, the rate of return on social security is the same as that on private saving. We can now derive the intertemporal budget constraint. From equation (4),

(19) $$S_t = C_{2,t+1}/(1+r_{t+1}) - T.$$

Substituting equation (19) into equation (2) yields

(20) $$C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = Aw_t - T + T,$$

or simply

(21) $$C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = Aw_t.$$

This is just the usual intertemporal budget constraint in the Diamond model. Solving the individual's maximization problem yields the usual Euler equation:

(22) $$C_{2,t+1} = \left[1/(1+\rho)\right](1+r_{t+1})C_{1,t}$$

Substituting this into the budget constraint, equation (21), yields

(23) $$C_{1,t} = \left[(1+\rho)/(2+\rho)\right]Aw_t.$$

To get saving per person, substitute equation (23) into equation (2):

(24) $$S_t = Aw_t - \left[(1+\rho)/(2+\rho)\right]Aw_t - T,$$

or simply

(25) $$S_t = \left[1/(2+\rho)\right]Aw_t - T.$$

The social security tax causes a one-for-one reduction in private saving.

The capital stock in period t +1 will be equal to the sum of total private saving of the young plus the total amount invested by the government. Hence

(26) $K_{t+1} = S_t L_t + TL_t$.

Dividing both sides of (26) by AL_{t+1} to convert this into units of effective labor, and using equation (25) yields

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(27) $k_{t+1} = \left(\frac{1}{1+n}\right)\left[\left(\frac{1}{2+\rho}\right)w_t - \frac{T}{A}\right] + \left(\frac{1}{1+n}\right)\frac{T}{A}$,

which simplifies to

(28) $k_{t+1} = \left[1/(1+n)\right]\left[1/(2+\rho)\right] w_t$.

Using equation (14) to substitute for the wage yields

(29) $k_{t+1} = \left[1/(1+n)\right]\left[1/(2+\rho)\right](1-\alpha)k_t^{\alpha}$.

Thus the fully-funded social security system has no effect on the relationship between the capital stock in successive periods.

(b) (ii) Since there is no effect on the relationship between k_{t+1} and k_t, the balanced-growth-path value of k is the same as it was before the introduction of the fully-funded social security system. (Note that we have been assuming that the amount of the tax is not greater than the amount of saving each individual would have done in the absence of the tax). The basic idea is that total investment and saving is still the same each period; the government is simply doing some of the saving for the young. Since social security pays the same rate of return as private saving, individuals are indifferent as to who does the saving. Thus individuals offset one-for-one any saving that the government does for them.

Problem 2.18

(a) In the decentralized equilibrium, there will be no intergenerational trade. Even if the young would like to trade goods this period for goods next period, the only people around to trade with are the old. Unfortunately, the old will be dead – and thus in no position to complete the trade – next period.

The individual's utility function is given by

(1) $\ln C_{1,t} + \ln C_{2,t+1}$.

The constraints are

(2) $C_{1,t} + F_t = A$, and (3) $C_{2,t+1} = xF_t$,

where F_t is the amount stored by the individual.

Substituting equation (3) into (2) yields the intertemporal budget constraint:

(4) $C_{1,t} + C_{2,t+1}/x = A$.

The individual's problem is to maximize lifetime utility, as given by equation (1), subject to the intertemporal budget constraint, as given by equation (4). Set up the Lagrangian:

(5) $\mathcal{L} = \ln C_{1,t} + \ln C_{2,t+1} + \lambda\left[A - C_{1,t} - C_{2,t+1}/x\right]$.

The first-order conditions are given by

(6) $\partial\mathcal{L}/\partial C_{1,t} = 1/C_{1,t} - \lambda = 0 \Rightarrow 1/C_{1,t} = \lambda$,

and

(7) $\partial\mathcal{L}/\partial C_{2,t+1} = 1/C_{2,t+1} - \lambda/x = 0 \Rightarrow 1/C_{2,t+1} = \lambda/x$.

Substitute equation (6) into equation (7) and rearrange to obtain

(8) $C_{2,t+1} = xC_{1,t}$.

Substitute equation (8) into the intertemporal budget constraint, equation (4), to obtain

(9) $C_{1,t} + xC_{1,t}/x = A$,

or simply

(10) $C_{1,t} = A/2$.

To obtain an expression for second-period consumption, substitute equation (10) into equation (8):

(11) $C_{2,t+1} = xA/2$.

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When young, each individual consumes half of her endowment and stores the other half; that is, $f_t = 1/2$. This allows her to consume xA/2 when old. Note that with log utility, the fraction of her endowment that the individual stores does not depend upon the return to storage.

(b) What is consumption per unit of effective labor at time t? First, calculate total consumption at time t:

(12) $C_t = C_{1,t}L_t + C_{2,t}L_{t-1}$,

where there are L_t young and L_{t-1} old individuals alive at time t. Each young person consumes the fraction of her endowment that she does not store, (1 - f)A, and each old person gets to consume the gross return on the fraction of her endowment that she stored, fxA. Thus

(13) $C_t = (1-f)AL_t + fxAL_{t-1}$.

To convert this into units of time t effective labor, divide both sides by AL_t to get

(14) $C_t/AL_t = (1-f) + f\left[x/(1+n)\right]$.

Thus consumption per unit of time t effective labor is a weighted average of one and something less than one, since x < (1 + n). It will therefore be maximized when the weight on one is one; that is, when f = 0. (We could also carry out this analysis on consumption per person alive at time t which would not change the result here).

The decentralized equilibrium, with f = 1/2, is not Pareto efficient. Since intergenerational trade is not possible, individuals are "forced" into storage because that is the only way they can save and consume in old age. They must do this even if the return on storage, x, is low. However, at any point in time, a social planner could take one unit from each young person and give (1 + n) units to each old person since there are fewer old persons than young persons. With (1 + n) > x, this gives a better return than storage. Therefore, the social planner could raise welfare by taking the half of each generation's endowment that it was going to store and instead give it to the old. The planner could then do this each period. This allows individuals to consume A/2 units when young – the same as in the decentralized equilibrium – but it also allows them to consume (1 + n)A/2 units when old. This is greater than the xA/2 units of consumption when old that they would have had in the decentralized equilibrium with storage.

Problem 2.19

(a) The individual has a utility function given by

(1) $\ln C_{1,t} + \ln C_{2,t+1}$,

and constraints, expressed in units of money, given by

(2) $P_tC_{1,t} = P_tA - P_tF_t - M_t^d$, and

(3) $P_{t+1}C_{2,t+1} = P_{t+1}xF_t + M_t^d$,

where M_t^d is nominal money demand and F_t is the amount stored.

One way of thinking about the problem is the following. The individual has two decisions to make. The first is to decide how much of her endowment to consume and how much to "save". Then she must decide the way in which to save, through storage, by holding money or a combination of both. With log utility, we can separate the two decisions since the rate of return on "saving" will not affect the fraction of the first-period endowment that is saved. From the solution to Problem 2.18, we know she will consume half of the endowment in the first period, regardless of the rate of return on money or storage. Thus

(4) $C_{1,t} = A/2$.

What does she do with the other half? That will depend upon the gross rate of return on storage, x, relative to the gross rate of return on money, which is P_t/P_{t+1} . The gross rate of return on money is P_t/P_{t+1}

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since the individual can sell one unit of consumption in period t and get P_t units of money. In period t + 1, one unit of consumption costs P_{t+1} units of money and thus one unit of money will buy $1/P_{t+1}$ units of consumption. Thus the individual's P_t units of money will buy P_t/P_{t+1} units of consumption in period t+1.

CASE 1 : $x > P_t/P_{t+1}$
She will consume half of her endowment, store the rest and not hold any money since the rate of return on money is less than the rate of return on storage. Thus

$$C_{1,t} = A/2 \qquad F_t = A/2 \qquad M_t^d/P_t = 0 \qquad C_{2,t+1} = xA/2.$$

CASE 2 : $x < P_t/P_{t+1}$
Now storage is dominated by holding money. She will consume half of her endowment and then sell the rest for money:

$$C_{1,t} = A/2 \qquad F_t = 0 \qquad M_t^d/P_t = A/2 \qquad C_{2,t+1} = \left[P_t/P_{t+1}\right]\left[A/2\right].$$

CASE 3 : $x = P_t/P_{t+1}$
Money and storage pay the same rate of return. She will consume half of her endowment and is then indifferent as to how much of the other half to store and how much of it to sell for money. Let $\alpha \in [0,1]$ be the fraction of saving that is in the form of money. Thus

$$C_{1,t} = A/2 \qquad F_t = (1-\alpha)A/2 \qquad M_t^d/P_t = \alpha A/2 \qquad C_{2,t+1} = xA/2 = \left[P_t/P_{t+1}\right]\left[A/2\right].$$

(b) Equilibrium requires that aggregate real money demand equal aggregate real money supply.
We can derive expressions for both real money demand and supply in period t:
aggregate real money demand = $L_t\left[A/2\right]$, and
aggregate real money supply = $\left[L_0/(1+n)\right]M/P_t = \left[L_t/(1+n)^{t+1}\right]M/P_t$.
The expression for aggregate real money supply uses the fact that in period 0, each old person, and there are $[L_0/(1 + n)]$ of them, receives M units of money. The last step then uses the fact that since population grows at rate n, $L_t = (1 + n)^t L_0$ and thus $L_0 = L_t/(1 + n)^t$.
We can then use the equilibrium condition to solve for P_t :

$$L_t\left[A/2\right] = \left[L_t/(1+n)^{t+1}\right]M/P_t \qquad \Rightarrow \qquad P_t = 2M/\left[A(1+n)^{t+1}\right]. \quad (5)$$

We can similarly derive expressions for real money demand and supply in period t + 1:
aggregate real money demand = $L_{t+1}\left[A/2\right] = (1+n)\,L_t\left[A/2\right]$, and
aggregate real money supply = $\left[L_t/(1+n)^{t+1}\right]M/P_{t+1}$.
We can then use the equilibrium condition to solve for P_{t+1} :

$$(1+n)\,L_0\left[A/2\right] = \left[L_t/(1+n)^{t+1}\right]M/P_{t+1} \qquad \Rightarrow \qquad P_{t+1} = 2M/\left[A(1+n)^{t+2}\right]. \quad (6)$$

Dividing equation (6) by equation (5) yields

$$P_{t+1}/P_t = 1/(1+n) \qquad \Rightarrow \qquad P_{t+1} = P_t/(1+n).$$

This analysis holds for all time periods $t \geq 0$ and so $P_{t+1} = P_t/(1 + n)$ is an equilibrium. This shows that if money is introduced into a dynamically inefficient economy, storage will not be used. The monetary equilibrium will thus result in attainment of the "golden-rule" level of storage. See the solution to part (b) of Problem 2.18 for an explanation of the reason that zero storage maximizes consumption per unit of effective labor.

(c) This is the situation where $P_t/P_{t+1} = x$; the return on money is equal to the return on storage. In this case, individuals are indifferent as to how much of their saving to store and how much to hold in the form

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of money. Let $\alpha_t \in [0,1]$ be the fraction of saving held in the form of money in period t. We can again derive expressions for aggregate real money demand and supply in period t:

aggregate real money demand = $L_t\alpha_t[A/2]$, and

aggregate real money supply = $[L_0/(1+n)]M/P_t = [L_t/(1+n)^{t+1}]M/P_t$.

We can then use the equilibrium condition to solve for P_t:

$$L_t\alpha_t[A/2] = [L_t/(1+n)^{t+1}]M/P_t \quad \Rightarrow \quad P_t = 2M/[\alpha_t A(1+n)^{t+1}]. \quad (7)$$

We can similarly derive expressions for real money demand and supply in period t + 1:

aggregate real money demand = $L_{t+1}\alpha_{t+1}[A/2] = (1+n)L_t\alpha_{t+1}[A/2]$, and

aggregate real money supply = $[L_t/(1+n)^{t+1}]M/P_{t+1}$.

We can then use the equilibrium condition to solve for P_{t+1}:

$$(1+n)L_0\alpha_{t+1}[A/2] = [L_t/(1+n)^{t+1}]M/P_{t+1} \quad \Rightarrow \quad P_{t+1} = 2M/[\alpha_{t+1}A(1+n)^{t+2}]. \quad (8)$$

Dividing equation (8) by (7) yields

$$P_{t+1}/P_t = [\alpha_t/\alpha_{t+1}][1/(1+n)].$$

For $P_{t+1}/P_t = 1/x$, we need

$$[\alpha_t/\alpha_{t+1}][1/(1+n)] = 1/x \quad \Rightarrow \quad [\alpha_{t+1}/\alpha_t] = [x/(1+n)] < 1.$$

Thus for all $t \geq 0$, $P_{t+1} = P_t/x$ will be an equilibrium for any path of α's that satisfies $\alpha_{t+1}/\alpha_t = x/(1+n)$.

(d) $P_t = \infty$ – money is worthless – is also an equilibrium. This occurs if the young generation at time 0 does not believe that money will be valued in the next period and thus that the generation one individuals will not accept money for goods. In that case, in period 0, the young simply consume half of their endowment and store the rest, and the old have some useless pieces of paper to go along with their endowment. This is an equilibrium with real money demand equal to zero and real money supply equal to zero as well. If no one believes the next generation will accept money for goods, this equilibrium continues for all future time periods.

This will be the only equilibrium if the economy ends at some date T. The young at date T will not want to sell any of their endowment. They will maximize the utility of their one-period life by consuming all their endowment in period T. Thus, if the old at date T held any money, they would be stuck with it and it would be useless to them. Thus when they are young, in period T - 1, they will not sell any of their endowment for money, knowing that the money will be of no use to them when old. Thus, if the old at date T - 1 held any money, they would be stuck with it and it would be useless to them. Thus the old at T - 1 will not want any money when they are young and so on. Working backward, no one would ever want to sell goods for money and money would not be valued.

Problem 2.20

(a) (i) The individual has a utility function given by

(1) $U = \ln C_{1,t} + \ln C_{2,t+1}$,

and a lifetime budget constraint given by

(2) $Q_t C_{1,t} + Q_{t+1} C_{2,t+1} = Q_t (A - S_t) + Q_{t+1} x S_t$.

From Problem 2.18, we know that with log utility, the individual wants to consume A/2 in the first period. The way in which the individual accomplishes this depends on the gross rate of return on storage, x, relative to the gross rate of return on trading.

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The individual can sell one unit of the good in period t for Q_t . In period t + 1, it costs Q_{t+1} to obtain one unit of the good or equivalently, it costs one to obtain $1/Q_{t+1}$ units of the good. Thus for Q_t , it is possible to obtain Q_t /Q_{t+1} units of the good. Thus selling a unit of the good in period t allows the individual to buy Q_t /Q_{t+1} units of the good in period t + 1. Thus the gross rate of return on trading is Q_t /Q_{t+1} .

Now, $Q_{t+1} = Q_t /x$ for all $t > 0$ is equivalent to $x = Q_t / Q_{t+1}$ for all $t > 0$. In other words, the rate of return on storage is equal to the rate of return on trading and hence the individual is indifferent as to the amount to store and the amount to trade. Let $\alpha_t \in [0, 1]$ represent the fraction of "saving", A/2, that the individual sells in period t. That is, the individual sells $\alpha_t (A/2)$ in period t. This allows the individual to buy the amount $\alpha_t (Q_t /Q_{t+1})(A/2)$ when she is old in period t + 1. The individual stores a fraction $(1 - \alpha_t)$ of her "saving". Thus

(3) $S_t = (1 - \alpha_t)(A/2)$.

Consumption in period t + 1 will be equal to the amount the individual buys plus the amount she has through storage. Thus

(4) $C_{2,t+1} = \alpha_t (Q_t /Q_{t+1})(A/2) + (1 - \alpha_t)x(A/2)$.

Since we are considering a case in which $Q_t /Q_{t+1} = x$, equation (4) can be rewritten as

(5) $C_{2,t+1} = \alpha_t x(A/2) + (1 - \alpha_t)x(A/2) = x(A/2)$.

Consider some period t + 1 and let L represent the total number of individuals born each period, which is constant. Aggregate supply in period t + 1 is equal to the total number of young individuals, L, multiplied by the amount that each young individual wishes to sell, $\alpha_{t+1} (A/2)$. Thus

(6) $\text{Aggregate Supply}_{t+1} = L\alpha_{t+1} (A/2)$.

Aggregate demand in period t + 1 is equal to the total number of old individuals, L, multiplied by the amount each old individual wishes to buy, $(Q_t /Q_{t+1})\alpha_t (A/2)$. Thus

(7) $\text{Aggregate Demand}_{t+1} = L(Q_t /Q_{t+1})\alpha_t (A/2)$.

For the market to clear, aggregate supply must equal aggregate demand or

(8) $L\alpha_{t+1} (A/2) = L(Q_t /Q_{t+1})\alpha_t (A/2)$,

or simply

(9) $\alpha_{t+1} = (Q_t /Q_{t+1})\alpha_t$.

Since the proposed price path has $Q_{t+1} = Q_t /x$, the equilibrium condition given by equation (9) can also be written as

(10) $\alpha_{t+1} = x\alpha_t$.

Now consider the situation in period 0. The old individuals simply consume their endowment. Thus we must have α_0 equal to zero in order for the market to clear in period 0. Thus equation (10) implies that we must have $\alpha_t = 0$ for all $t \geq 0$.

The resulting equilibrium is the same as that in part (a) of Problem 2.18. The individual consumes half of her endowment in the first period of life, stores the rest and consumes xA/2 in the second period of life. Note that with $x < 1 + n$ here (since $n = 0$ and $x < 1$), this equilibrium is dynamically inefficient. Thus eliminating incomplete markets by allowing individuals to trade before the start of time does not eliminate dynamic inefficiency.

(a) (ii) Suppose the auctioneer announces $Q_{t+1} < Q_t /x$ or equivalently $x < (Q_t /Q_{t+1})$ for some date t. This means that trading dominates storage for the young at date t. This means that the young at date t will want to sell all their saving – $\alpha_t = 1$ so that they want to sell A/2 – and not store anything. Thus aggregate supply in period t is equal to L(A/2). For the old at date t, Q_{t+1} is irrelevant. They based their decision of how much to buy when old on Q_t / Q_{t-1} which was equal to x. Thus as described in part (a) (i), old individuals were not planning to buy anything. Thus aggregate demand in period t is zero. Thus aggregate demand will be less than aggregate supply and the market for the good will not clear. Thus the proposed price path cannot be an equilibrium.

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Suppose instead that the auctioneer announces $Q_{t+1} > Q_t/x$ or equivalently $x > (Q_t/Q_{t+1})$ for some date t. This means that storage dominates trading for the young at date t. This means that the young at date t will want to store their entire endowment and will want to buy A/2. For the old at date t, Q_{t+1} is irrelevant. They based their decision of how much to trade when old on Q_t/Q_{t-1} which was equal to x. Thus each old individual was not planning to buy or sell anything. Thus aggregate demand exceeds aggregate supply and the market for the good will not clear. Thus the proposed price path cannot be an equilibrium.

(b) Consider the social planner's problem. The planner can divide the resources available for consumption between the young and the old in any matter. The planner can take, for example, one unit of each young person's endowment and transfer it to the old. Since there are the same number of old and young people in this model, this increases the consumption of each old person by one. With $x < 1$, this method of transferring from the young to the old provides a better return than storage. If the economy did not end at some date T, the planner could prevent this change from making anyone worse off by requiring the next generation of young to make the same transfer in the following period. However, if the economy ends at some date T, the planner cannot do this. Taking anything from the young at date T would make them worse off since the planner cannot give them anything in return the next period; there is no next period. Thus the planner cannot make some generations better off without making another generation worse off. Thus the decentralized equilibrium is Pareto-efficient.

(c) It is infinite duration that is the source of the dynamic inefficiency. Allowing individuals to trade before the start of time requires a price path that results in an equilibrium which is equivalent to the situation where such a market does not exist. This equilibrium is not Pareto-efficient; a social planner could raise welfare by doing the procedure described in part (b). However, removing infinite duration also removes the social planner's ability to Pareto improve the decentralized equilibrium, as explained in part (b).

Problem 2.21

(a) The individual has utility function given by

(1) $\dfrac{C_{1,t}^{1-\theta}}{1-\theta} + \dfrac{C_{2,t+1}^{1-\theta}}{1-\theta} \qquad \theta < 1.$

The constraints expressed in units of money are

(2) $P_t C_{1,t} = P_t A - M_t^d,$ and (3) $P_{t+1} C_{2,t+1} = M_t^d.$

Combining equations (2) and (3) yields the lifetime budget constraint:

(4) $P_t C_{1,t} + P_{t+1} C_{2,t+1} = P_t A.$

Note that $\theta < 1$ means that the elasticity of substitution, $1/\theta$, is greater than one. Thus when the rate of return on saving increases, the substitution effect dominates; the individual will consume less now and save more. This is essentially what we will be showing here. As the rate of return on holding money, which is P_t/P_{t+1}, rises, the individual wishes to hold more money.

The individual's problem is to maximize (1) subject to (4). Set up the Lagrangian:

(5) $\mathcal{L} = \dfrac{C_{1,t}^{1-\theta}}{1-\theta} + \dfrac{C_{2,t+1}^{1-\theta}}{1-\theta} + \lambda\left[P_t A - P_t C_{1,t} - P_{t+1} C_{2,t+1}\right]$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial C_{1,t}} = C_{1,t}^{-\theta} - \lambda P_t = 0 \Rightarrow \lambda = \frac{C_{1,t}^{-\theta}}{P_t}, \text{ and} \qquad (6)$$

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$$\frac{\partial \mathcal{L}}{\partial C_{2,t+1}} = C_{2,t+1}^{-\theta} - \lambda P_{t+1} = 0 \Rightarrow C_{2,t+1}^{-\theta} = \lambda P_{t+1}. \quad (7)$$

Substitute (6) into (7) to obtain

(8) $C_{2,t+1}^{-\theta} = C_{1,t}^{-\theta} P_{t+1} / P_t$,

Taking both sides of equation (8) to the exponent $-1/\theta$ gives us

(9) $C_{2,t+1} = \left(P_t / P_{t+1}\right)^{1/\theta} C_{1,t}$.

This is the Euler equation, which can now be substituted into the budget constraint given by equation (4):

(10) $P_t C_{1,t} + P_{t+1}\left(P_t / P_{t+1}\right)^{1/\theta} C_{1,t} = P_t A$.

Dividing by P_t yields

(11) $C_{1,t} + \left(P_{t+1}/P_t\right)\left(P_t / P_{t+1}\right)^{1/\theta} C_{1,t} = A$,

and simplifying yields

(12) $C_{1,t} + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta} C_{1,t} = A$

or

(13) $C_{1,t}\left[1 + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}\right] = A$.

Thus consumption when young is given by

(14) $C_{1,t} = \dfrac{A}{1 + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}}$.

To get the amount of her endowment that the individual sells for money (in real terms), we can use equation (2), expressed in real terms

(15) $M_t^d / P_t = A - C_{1,t}$.

Substitute equation (14) into (15) to obtain

(16) $\dfrac{M_t^d}{P_t} = A - \dfrac{A}{1 + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}}$,

which can be rewritten as

(17) $\dfrac{M_t^d}{P_t} = A\left[1 - \dfrac{1}{1 + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}}\right]$.

Simplifying by getting a common denominator yields

(18) $\dfrac{M_t^d}{P_t} = A \dfrac{\left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}}{1 + \left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}}$.

Dividing the top and bottom of the right-hand side of equation (18) by $\left(P_t / P_{t+1}\right)^{(1-\theta)/\theta}$ yields

(19) $\dfrac{M_t^d}{P_t} = \dfrac{A}{\left(P_t / P_{t+1}\right)^{(\theta-1)/\theta} + 1}$.

Thus the fraction of her endowment that the individual sells for money is

(20) $h_t = \dfrac{1}{\left(P_t / P_{t+1}\right)^{(\theta-1)/\theta} + 1}$.

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Taking the derivative of h_t with respect to (P_t/P_{t+1}) allows us to shows that the fraction of her endowment that the agent sells for money is an increasing function of the rate of return on holding money:

$$(21)\quad \frac{\partial h_t}{\partial\left(P_t/P_{t+1}\right)} = \frac{-\left[(\theta-1)/\theta\right]\left(P_t/P_{t+1}\right)^{\left[(\theta-1)/\theta\right]-1}}{\left[\left(P_t/P_{t+1}\right)^{(\theta-1)/\theta}+1\right]^2} > 0 \quad \text{for } \theta < 1.$$

We can also show that as the rate of return on money goes to zero, the amount of her endowment that the individual sells for money goes to zero. Use equation (18) to rewrite h_t as

$$(22)\quad h_t = \frac{\left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}}{\left[1+\left(P_t/P_{t+1}\right)^{(1-\theta)/\theta}\right]},$$

and thus

$$(23)\quad \lim_{\left(P_t/P_{t+1}\right)\to 0} h_t = 0/(1+0) = 0.$$

(b) The constraints expressed in real terms are

$$(24)\quad C_{1,t} = A - M_t^d/P_t, \qquad \text{and} \qquad (25)\quad C_{2,t+1} = M_t^d/P_{t+1}.$$

Since there is no population growth, we can normalize the population to one without loss of generality. From (25), a generation born at time t plans to buy M^d_t/P_{t+1} units of the good when it is old. Thus, the generation born at time 0 plans to buy M^d_0/P_1 units when it is old (in period 1). Use equation (19) to find M^d_0, substituting t = 0:

$$(26)\quad M_0^d = \frac{P_0 A}{\left(P_0/P_1\right)^{(\theta-1)/\theta}+1},$$

which, expressed in real terms, is

$$(27)\quad \frac{M_0^d}{P_1} = \frac{\left[P_0/P_1\right]A}{\left(P_0/P_1\right)^{(\theta-1)/\theta}+1}.$$

From equation (24), a generation born at time t plans to sell M^d_t/P_t units of the good for money. Thus the generation born at time 1 plans to sell M^d_1/P_1 units of the good. Substituting t = 1 into equation (19) gives us

$$(28)\quad \frac{M_1^d}{P_1} = \frac{A}{\left(P_1/P_2\right)^{(\theta-1)/\theta}+1}.$$

For the amount of the consumption good that generation 0 wishes to buy with its money, given by equation (27), to be equal to the amount of the consumption good that generation 1 wishes to sell for money, given by equation (28), we need

$$(29)\quad \frac{\left[P_0/P_1\right]A}{\left(P_0/P_1\right)^{(\theta-1)/\theta}+1} = \frac{A}{\left(P_1/P_2\right)^{(\theta-1)/\theta}+1},$$

or

$$(30)\quad \frac{\left(P_0/P_1\right)^{(\theta-1)/\theta}+1}{\left(P_1/P_2\right)^{(\theta-1)/\theta}+1} = \frac{P_0}{P_1}.$$

Now with $P_0/P_1 < 1$, this means that we need

$$(31)\quad \left(P_0/P_1\right)^{(\theta-1)/\theta}+1 < \left(P_1/P_2\right)^{(\theta-1)/\theta}+1,$$

which is equivalent to

$$(32)\quad \left(P_0/P_1\right)^{(\theta-1)/\theta} < \left(P_1/P_2\right)^{(\theta-1)/\theta}.$$

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Since $(\theta - 1)/\theta$ is negative, this implies

(33) $$\frac{P_1}{P_2} < \frac{P_0}{P_1} < 1.$$

(c) Iterating this reasoning forward, the rate of return on money must be falling over time. That is,

$$1 > \frac{P_0}{P_1} > \frac{P_1}{P_2} > \frac{P_2}{P_3} > \ldots \qquad \text{and so } \frac{P_t}{P_{t+1}} \to 0.$$

As shown in part (a), this means that the fraction of the endowment that is sold for money will also go to zero. The economy approaches the situation in which individuals consume their entire endowment in the first period. This is an equilibrium path in the sense that every period, markets will clear. Each period, the real money demand by the young will be equal to real money supplied by the old and they will both go to zero as t gets large.

(d) If $P_0/P_1 > 1$, we obtain the opposite result for the path of prices. That is, P_t/P_{t+1} will rise over time:

$$1 < \frac{P_0}{P_1} < \frac{P_1}{P_2} < \frac{P_2}{P_3} < \ldots \qquad \text{and so } \frac{P_t}{P_{t+1}} \to \infty.$$

From equation (20) we can see that this means that the fraction of the endowment sold for money will go to one. In other words, the economy approaches the situation in which no one consumes anything in the first period and individuals sell their entire endowment for money. Thus, total real money demand will go to A, the endowment of the young (we have normalized the population to one). But with this path of prices, the price level goes to zero, which means that real money supplied by the old goes to infinity. Thus this cannot represent an equilibrium path for the economy because there will be a time period when real money supply will exceed real money demand and the market will not clear.

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