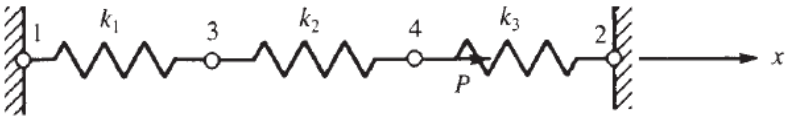
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Chapter 2

2.1

(a)



$$[k^{(1)}] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix}$$

$$[k_3{}^{(3)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}$$

$$[K] = [k^{(1)}] + [k^{(2)}] + [k^{(3)}]$$

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix}$$

(b) Nodes 1 and 2 are fixed so $u_1 = 0$ and $u_2 = 0$ and $[K]$ becomes

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ P \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\{F\} = [K]\{d\} \Rightarrow [K]^{-1}\{F\} = [K]^{-1}[K]\{d\}$$

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$$\Rightarrow [K]^{-1}\{F\} = \{d\}$$

Using the adjoint method to find $[K^{-1}]$

$$C_{11} = k_2 + k_3 \qquad C_{21} = (-1)^3(-k_2)$$

$$C_{12} = (-1)^{1+2}(-k_2) = k_2 \qquad C_{22} = k_1 + k_2$$

$$[C] = \begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix} \text{ and } C^T = \begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}$$

$$\det [K] = |[K]| = (k_1 + k_2)(k_2 + k_3) - (-k_2)(-k_2)$$

$$\Rightarrow \quad |[K]| = (k_1 + k_2)(k_2 + k_3) - k_2^2$$

$$[K^{-1}] = \frac{[C^T]}{\det K}$$

$$[K^{-1}] = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}}{(k_1 + k_2)(k_2 + k_3) - k_2^2} = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ P \end{Bmatrix}}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow \quad u_3 = \frac{k_2 P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow \quad u_4 = \frac{(k_1 + k_2) P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

(c) In order to find the reaction forces we go back to the global matrix $F = [K]\{d\}$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$F_{1x} = -k_1 u_3 = -k_1 \frac{k_2 P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow F_{1x} = \frac{-k_1 k_2 P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

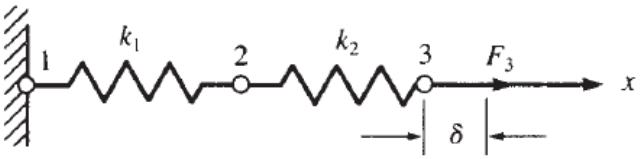
$$F_{2x} = -k_3 u_4 = -k_3 \frac{(k_1 + k_2) P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow F_{2x} = \frac{-k_3 (k_1 + k_2) P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

2.2

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$$k_1 = k_2 = k_3 = 100\ \frac{\text{N}}{\text{mm}} = 1000\ \frac{\text{N}}{\text{cm}}$$

$$[k^{(1)}] = \begin{matrix} \\ (1) \\ (2) \end{matrix}\begin{matrix} \begin{matrix} (1) & (2) \end{matrix} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{matrix}; \quad [k^{(2)}] = \begin{matrix} \\ (2) \\ (3) \end{matrix}\begin{matrix} \begin{matrix} (2) & (3) \end{matrix} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{matrix}$$

By the method of superposition the global stiffness matrix is constructed.

$$[K] = \begin{matrix} \\ (1) \\ (2) \\ (3) \end{matrix}\begin{matrix} \begin{matrix} (1) & (2) & (3) \end{matrix} \\ \begin{bmatrix} k & -k & 0 \\ -k & k+k & -k \\ 0 & -k & k \end{bmatrix} \end{matrix} \Rightarrow [K] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

Node 1 is fixed $\Rightarrow u_1 = 0$ and $u_3 = \delta$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = ? \end{Bmatrix} = \begin{bmatrix} k & k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = \delta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 \\ \delta \end{Bmatrix} \Rightarrow \begin{Bmatrix} 0 = 2k\,u_2 - k\delta \\ F_{3x} = -k\,u_2 + k\delta \end{Bmatrix}$$

$$\Rightarrow \quad u_2 = \frac{k\delta}{2k} = \frac{\delta}{2} = \frac{20\text{ mm}}{2} \Rightarrow u_2 = 10\text{ mm} = 1\text{ cm}$$

$$F_{3x} = -k\,(1\text{ cm}) + k\,(2\text{ cm})$$

$$F_{3x} = (-1000\ \frac{\text{N}}{\text{cm}})\,(1\text{ cm}) + (1000\ \frac{\text{N}}{\text{cm}})\,(2\text{ cm})$$

$$F_{3x} = 1000\text{ N}$$

Internal forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 1\text{ cm} \end{Bmatrix}$$

$$\Rightarrow \quad f_{1x}^{(1)} = (-1000\ \frac{\text{N}}{\text{cm}})\,(1\text{ cm}) \Rightarrow f_{1x}^{(1)} = -1000\text{ N}$$

$$f_{2x}^{(1)} = (1000\ \frac{\text{N}}{\text{cm}})\,(1\text{ cm}) \Rightarrow f_{2x}^{(1)} = 1000\text{ N}$$

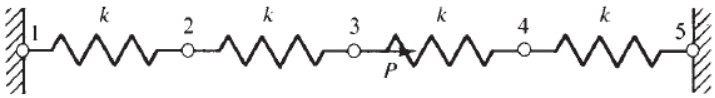
Element (2)

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$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 = 1\text{ cm} \\ u_3 = 2\text{ cm} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -1000\text{ N} \\ f_{3x}^{(2)} = 1000\text{ N} \end{matrix}$$

2.3



(a) $[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

By the method of superposition we construct the global $[K]$ and knowing $\{F\} = [K]\{d\}$ we have

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = P \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 = 0 \end{Bmatrix}$$

(b) $$\begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \Rightarrow \begin{matrix} 0 = 2ku_2 - ku_3 & (1) \\ P = -ku_2 + 2ku_3 - ku_4 & (2) \\ 0 = -ku_3 + 2ku_4 & (3) \end{matrix}$$

$$\Rightarrow \quad u_2 = \frac{u_3}{2} \; ; u_4 = \frac{u_3}{2}$$

Substituting in the second equation above

$$P = -k\,u_2 + 2k\,u_3 - k\,u_4$$

$$\Rightarrow \quad P = -k\left(\frac{u_3}{2}\right) + 2k\,u_3 - k\left(\frac{u_3}{2}\right)$$

$$\Rightarrow \quad P = k\,u_3$$

$$\Rightarrow \quad u_3 = \frac{P}{k}$$

$$u_2 = \frac{P}{2k} \; ; u_4 = \frac{P}{2k}$$

(c) In order to find the reactions at the fixed nodes 1 and 5 we go back to the global equation $\{F\} = [K]\{d\}$

$$F_{1x} = -k\,u_2 = -k\,\frac{P}{2k} \Rightarrow F_{1x} = -\frac{P}{2}$$

$$F_{5x} = -k\,u_4 = -k\,\frac{P}{2k} \Rightarrow F_{5x} = -\frac{P}{2}$$

Check

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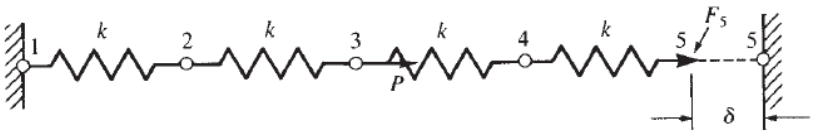
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$$\Sigma F_x = 0 \Rightarrow F_{1x} + F_{5x} + P = 0$$

$$\Rightarrow -\frac{P}{2} + \left(-\frac{P}{2}\right) + P = 0$$

$$\Rightarrow 0 = 0$$

2.4



(a) $[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

By the method of superposition the global $[K]$ is constructed.

Also $\{F\} = [K]\{d\}$ and $u_1 = 0$ and $u_5 = \delta$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 0 \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = \delta \end{Bmatrix}$$

(b) $0 = 2k\,u_2 - k\,u_3$ (1)

$0 = -ku_2 + 2k\,u_3 - k\,u_4$ (2)

$0 = -k\,u_3 + 2k\,u_4 - k\,\delta$ (3)

From (2)

$$u_3 = 2\,u_2$$

From (3)

$$u_4 = \frac{\delta + 2\,u_2}{2}$$

Substituting in Equation (2)

$$\Rightarrow -k\,(u_2) + 2k\,(2\,u_2) - k\left(\frac{\delta + 2u_2}{2}\right)$$

$$\Rightarrow -u_2 + 4\,u_2 - u_2 - \frac{\delta}{2} = 0 \Rightarrow u_2 = \frac{\delta}{4}$$

$$\Rightarrow u_3 = 2\,\frac{\delta}{4} \Rightarrow u_3 = \frac{\delta}{2}$$

$$\Rightarrow u_4 = \frac{\delta + 2\,\frac{\delta}{4}}{2} \Rightarrow u_4 = \frac{3\,\delta}{4}$$

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(c) Going back to the global equation

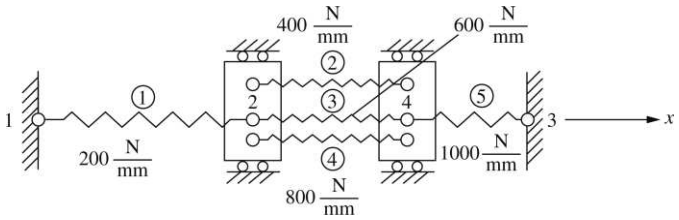
$$\{F\} = [K]\{d\}$$

$$F_{1x} = -k\ u_2 = -k\frac{\delta}{4} \Rightarrow F_{1x} = -\frac{k\,\delta}{4}$$

$$F_{5x} = -k\,u_4 + k\,\delta = -k\left(\frac{3\,\delta}{4}\right) + k\,\delta$$

$$\Rightarrow \quad F_{5x} = \frac{k\,\delta}{4}$$

2.5



$$[k^{(1)}] = \begin{matrix} \\ d_1 \\ d_2 \end{matrix}\begin{matrix} \begin{matrix} d_1 & d_2 \end{matrix} \\ \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \end{matrix}; \quad [k^{(2)}] = \begin{matrix} \\ d_2 \\ d_4 \end{matrix}\begin{matrix} \begin{matrix} d_2 & d_4 \end{matrix} \\ \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \end{matrix}$$

$$[k^{(3)}] = \begin{matrix} \\ d_2 \\ d_4 \end{matrix}\begin{matrix} \begin{matrix} d_2 & d_4 \end{matrix} \\ \begin{bmatrix} 600 & -600 \\ -600 & 600 \end{bmatrix} \end{matrix}; \quad [k^{(4)}] = \begin{matrix} \\ d_2 \\ d_4 \end{matrix}\begin{matrix} \begin{matrix} d_2 & d_4 \end{matrix} \\ \begin{bmatrix} 800 & -800 \\ -800 & 800 \end{bmatrix} \end{matrix}$$

$$[k^{(5)}] = \begin{matrix} \\ d_4 \\ d_3 \end{matrix}\begin{matrix} \begin{matrix} d_4 & d_3 \end{matrix} \\ \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \end{matrix}$$

Assembling global $[K]$ using direct stiffness method

$$[K] = \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 200+400+600+800 & 0 & -400-600-800 \\ 0 & 0 & 1000 & -1000 \\ 0 & -400-600-800 & -1000 & 400+600+800+1000 \end{bmatrix}$$

Simplifying

$$[K] = \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 2000 & 0 & -1800 \\ 0 & 0 & 1000 & -1000 \\ 0 & -1800 & -1000 & 2800 \end{bmatrix} \frac{\text{N}}{\text{mm}}$$

2.6 Now apply + 10,000 N at node 2 in spring assemblage of P 2.5.

$$\therefore \quad F_{2x} = 10{,}000 \text{ N}$$

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$$[K]\{d\} = \{F\}$$

$[K]$ from P 2.5

$$\begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 2000 & 0 & -1800 \\ 0 & 0 & 1000 & -1000 \\ 0 & -1800 & -1000 & 2800 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 10,000 \\ F_3 \\ 0 \end{Bmatrix} \quad \text{(A)}$$

where $u_1 = 0$, $u_3 = 0$ as nodes 1 and 3 are fixed.

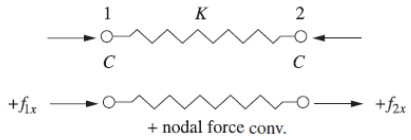
Using Equations (1) and (3) of (A)

$$\begin{bmatrix} 2000 & -1800 \\ -1800 & 2800 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 10,000 \\ 0 \end{Bmatrix}$$

Solving

$$u_2 = \ 11.86 \text{ mm}, \quad u_4 = 7.63 \text{ mm}$$

2.7



$$f_{1x} = C, \quad f_{2x} = -C$$

$$f = -k\delta = -k(u_2 - u_1)$$

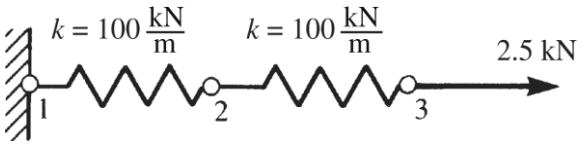
$$\therefore \quad f_{1x} = -k(u_2 - u_1)$$

$$f_{2x} = -(-k)(u_2 - u_1)$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore \quad [K] = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \text{ same as for tensile element}$$

2.8



$$k_1 = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \; k_2 = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

So

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$$[K] = 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\Rightarrow \begin{bmatrix} F_1 = ? \\ F_2 = 0 \\ F_3 = 2500 \end{bmatrix} = 10^5 \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$\Rightarrow \quad 0 = 2 \times 10^5 \, u_2 - 10^5 \, u_3 \tag{1}$$

$$2500 = -10^5 \, u_2 + 10^5 \, u_3 \tag{2}$$

From (1)

$$u_2 = \frac{10^5}{2 \times 10^5} \, u_3 \Rightarrow u_2 = 0.5 \, u_3 \tag{3}$$

Substituting (3) into (2)

$$\Rightarrow \quad 2500 = -10^5 (0.5 \, u_3) + 10^5 \, u_3$$

$$\Rightarrow \quad 2500 = 0.5 \times 10^5 \, u_3$$

$$\Rightarrow \quad u_3 = 0.05 \text{ m} = 5 \text{ cm}$$

$$\Rightarrow \quad u_2 = (0.5)(5 \text{ cm}) \Rightarrow u_2 = 2.5 \text{ cm}$$

Element 1–2

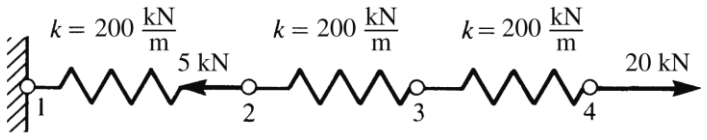
$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.025 \text{ m} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -2500 \text{ N} \\ f_{2x}^{(1)} = 2500 \text{ N} \end{matrix}$$

Element 2–3

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.025 \text{ m} \\ 0.05 \text{ m} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -2500 \text{ N} \\ f_{3x}^{(2)} = 2500 \text{ N} \end{matrix}$$

$$F_{1x} = 10^5 \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.025 \text{ m} \\ 0.05 \text{ m} \end{bmatrix} \Rightarrow F_{1x} = -2500 \text{ N}$$

2.9



$$[k^{(1)}] = 10^3 \begin{matrix} \\ (1) \\ (2) \end{matrix} \begin{matrix} \begin{matrix} (1) & (2) \end{matrix} \\ \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \end{matrix}$$

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$$[k^{(2)}] = 10^3 \begin{matrix} \\ (2) \\ (3) \end{matrix} \begin{bmatrix} (2) & (3) \\ 200 & -200 \\ -200 & 200 \end{bmatrix}$$

$$[k^{(3)}] = 10^3 \begin{matrix} \\ (3) \\ (4) \end{matrix} \begin{bmatrix} (3) & (4) \\ 200 & -200 \\ -200 & 200 \end{bmatrix}$$

$$[K] = 10^3 \begin{bmatrix} (1) & (2) & (3) & (4) \\ 200 & -200 & 0 & 0 \\ -200 & 400 & -200 & 0 \\ 0 & -200 & 400 & -200 \\ 0 & 0 & -200 & 200 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = -5000 \\ F_{3x} = 0 \\ F_{4x} = 20{,}000 \end{Bmatrix} = \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 400 & -200 & 0 \\ 0 & -200 & 400 & -200 \\ 0 & 0 & -200 & 200 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{aligned} u_1 &= 0 \\ u_2 &= 0.075 \text{ m} = 7.5 \text{ cm} \\ u_3 &= 0.175 \text{ m} = 17.5 \text{ cm} \\ u_4 &= 0.275 \text{ m} = 27.5 \text{ cm} \end{aligned}$$

Reactions

$$F_{1x} = 10^3 \ [200 \quad -200 \quad 0 \quad 0] \begin{Bmatrix} u_1 = & 0 \\ u_2 = & 0.075 \\ u_3 = & 0.175 \\ u_4 = & 0.275 \end{Bmatrix} \Rightarrow F_{1x} = -15000 \text{ N}$$

Element forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.075 \end{Bmatrix} \Rightarrow \begin{aligned} f_{1x}^{(1)} &= -15{,}000 \text{ N} \\ f_{2x}^{(1)} &= 15{,}000 \text{ N} \end{aligned}$$

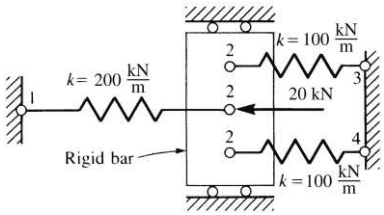
Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.075 \\ 0.175 \end{Bmatrix} \Rightarrow \begin{aligned} f_{2x}^{(2)} &= -20{,}000 \text{ N} \\ f_{3x}^{(2)} &= 20{,}000 \text{N} \end{aligned}$$

Element (3)

$$\begin{Bmatrix} f_{3x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.175 \\ 0.275 \end{Bmatrix} \Rightarrow \begin{aligned} f_{3x}^{(3)} &= -20{,}000 \\ f_{4x}^{(3)} &= 20{,}000 \end{aligned}$$

2.10



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$$[k^{(1)}] = 10^3 \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}$$

$$[k^{(2)}] = 10^3 \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix}$$

$$[k^{(3)}] = 10^3 \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = -20{,}000 \\ F_{3x} = ? \\ F_{4x} = ? \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 400 & -100 & -100 \\ 0 & -100 & 100 & 0 \\ 0 & -100 & 0 & 100 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \qquad u_2 = \frac{-20{,}000}{400 \times 10^3} = -0.05 \text{ m} = -5 \text{ cm}$$

Reactions

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 400 & -100 & -100 \\ 0 & -100 & 100 & 0 \\ 0 & -100 & 0 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.05 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 10{,}000 \\ -20{,}000 \\ 5000 \\ 5000 \end{Bmatrix} \text{ N}$$

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 10^3 \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.05 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 10{,}000 \\ -10{,}000 \end{Bmatrix} \text{ N}$$

Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 10^3 \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} -0.05 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -5000 \\ 5000 \end{Bmatrix} \text{ N}$$

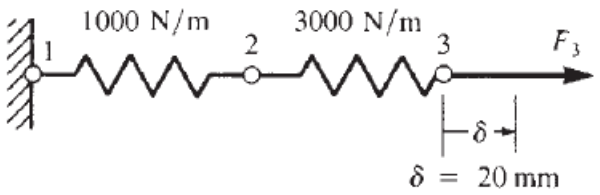
Element (3)

$$\begin{Bmatrix} f_{2x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = 10^3 \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} -0.05 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} \begin{Bmatrix} -5000 \\ 5000 \end{Bmatrix} \text{ N}$$

2.11

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$$[k^{(1)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}; \quad [k^{(2)}] = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = ? \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 4000 & -3000 \\ 0 & -3000 & 3000 \end{bmatrix} \begin{bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0.02 \text{ m} \end{bmatrix}$$

$$\Rightarrow \quad u_2 = 0.015 \text{ m}$$

Reactions

$$F_{1x} = (-1000)(0.015) \Rightarrow F_{1x} = -15 \text{ N}$$

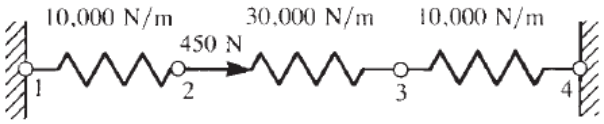
Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.015 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} -15 \\ 15 \end{Bmatrix} \text{ N}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{Bmatrix} 0.015 \\ 0.02 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{Bmatrix} -15 \\ 15 \end{Bmatrix} \text{ N}$$

2.12



$$[k^{(1)}] = [k^{(3)}] = 10000 \begin{Bmatrix} 1 & -1 \\ -1 & 1 \end{Bmatrix}$$

$$[k^{(2)}] = 10000 \begin{Bmatrix} 3 & -3 \\ -3 & 3 \end{Bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 450 \text{ N} \\ F_{3x} = 0 \\ F_{4x} = ? \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 4 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

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$$0 = -3u_2 + 4u_3 \Rightarrow u_2 = \frac{4}{3}u_3 \Rightarrow u_2 = 1.33\,u_3$$

$$450\text{ N} = 40000\,(1.33\,u_3) - 30000\,u_3$$

$$\Rightarrow \quad 450\text{ N} = (23200\ \frac{\text{N}}{\text{m}})\,u_3 \Rightarrow u_3 = 1.93 \times 10^{-2}\text{ m}$$

$$\Rightarrow \quad u_2 = 1.5\,(1.94 \times 10^{-2}) \Rightarrow u_2 = 2.57 \times 10^{-2}\text{ m}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 2.57 \times 10^{-2} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -257\text{ N} \\ f_{2x}^{(1)} = 257\text{ N} \end{matrix}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 30000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 2.57 \times 10^{-2} \\ 1.93 \times 10^{-2} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = 193\text{ N} \\ f_{3x}^{(2)} = -193\text{ N} \end{matrix}$$

Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.93 \times 10^{-2} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x}^{(3)} = 193\text{ N} \\ f_{4x}^{(3)} = -193\text{ N} \end{matrix}$$

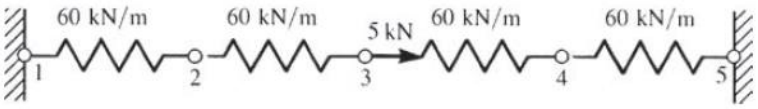
Reactions

$$\{F_{1x}\} = (10000\ \frac{\text{N}}{\text{m}})\,[1 \quad -1] \begin{Bmatrix} 0 \\ 2.57 \times 10^{-2} \end{Bmatrix} \Rightarrow F_{1x} = -257\text{ N}$$

$$\{F_{4x}\} = (10000\ \frac{\text{N}}{\text{m}})\,[-1 \quad 1] \begin{Bmatrix} 1.93 \times 10^{-2} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad F_{4x} = -193\text{ N}$$

2.13



$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 5\text{ kN} \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = 60 \begin{Bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{Bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = 0 \end{Bmatrix}$$

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$$0 = 2u_2 - u_3 \Rightarrow u_2 = 0.5\,u_3$$
$$0 = -u_3 + 2u_4 \Rightarrow u_4 = 0.5\,u_3 \Rightarrow u_2 = u_4$$

$$\Rightarrow \quad 5 \text{ kN} = -60\,u_2 + 120\,(2\,u_2) - 60\,u_2$$
$$\Rightarrow \quad 5 = 120\,u_2 \Rightarrow u_2 = 0.042 \text{ m}$$
$$\Rightarrow \quad u_4 = 0.042 \text{ m}$$
$$\Rightarrow \quad u_3 = 2(0.042) \Rightarrow u_3 = 0.084 \text{ m}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.042 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -2.5 \text{ kN} \\ f_{2x}^{(1)} = 2.5 \text{ kN} \end{matrix}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.042 \\ 0.084 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -2.5 \text{ kN} \\ f_{3x}^{(2)} = 2.5 \text{ kN} \end{matrix}$$

Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.084 \\ 0.042 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x}^{(3)} = 2.5 \text{ kN} \\ f_{4x}^{(3)} = -2.5 \text{ kN} \end{matrix}$$

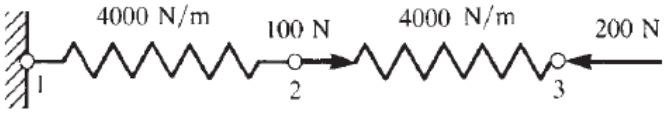
Element (4)

$$\begin{Bmatrix} f_{4x} \\ f_{5x} \end{Bmatrix} = 60 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.042 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{4x}^{(4)} = 2.5 \text{ kN} \\ f_{5x}^{(4)} = -2.5 \text{ kN} \end{matrix}$$

$$F_{1x} = 60 \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.042 \end{Bmatrix} \Rightarrow F_{1x} = -2.5 \text{ kN}$$

$$F_{5x} = 60 \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.042 \\ 0 \end{Bmatrix} \Rightarrow F_{5x} = -2.5 \text{ kN}$$

2.14



$$[k^{(1)}] = [k^{(2)}] = 4000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 100 \\ F_{3x} = -200 \end{Bmatrix} = 4000 \left[\begin{array}{ccc} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{array}\right] \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$100 = 8000\,u_2 - 4000\,u_3$$
$$-200 = -4000\,u_2 + 4000\,u_3$$

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$$-100 = 4000\,u_2 \Rightarrow u_2 = -0.025 \text{ m}$$

$$100 = 8000\,(-0.025) - 4000\,u_3 \Rightarrow u_3 = -0.075 \text{ m}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 4000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.025 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = 100 \text{ N} \\ f_{2x}^{(1)} = -100 \text{ N} \end{matrix}$$

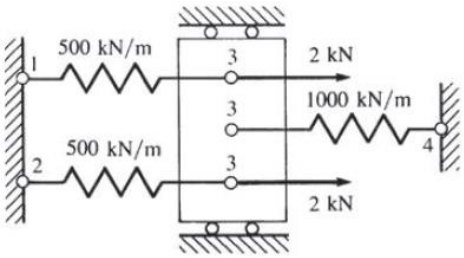
Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 4000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -0.025 \\ -0.075 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = 200 \text{ N} \\ f_{3x}^{(2)} = -200 \text{ N} \end{matrix}$$

Reaction

$$\{F_{1x}\} = 4000\,[1 \quad -1] \begin{Bmatrix} 0 \\ -0.025 \end{Bmatrix} \Rightarrow F_{1x} = 100 \text{ N}$$

2.15



$$[k^{(1)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix};\ [k^{(2)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix};\ [k^{(3)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = ? \\ F_{3x} = 4 \text{ kN} \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 500 & 0 & -500 & 0 \\ 0 & 500 & -500 & 0 \\ -500 & -500 & 2000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 0 \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_3 = 0.002 \text{ m}$$

Reactions

$$F_{1x} = (-500)\,(0.002) \Rightarrow F_{1x} = -1.0 \text{ kN}$$

$$F_{2x} = (-500)\,(0.002) \Rightarrow F_{2x} = -1.0 \text{ kN}$$

$$F_{4x} = (-1000)\,(0.002) \Rightarrow F_{4x} = -2.0 \text{ kN}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.002 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = \begin{Bmatrix} -1.0 \text{ kN} \\ 1.0 \text{ kN} \end{Bmatrix}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.002 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{Bmatrix} -1.0 \text{ kN} \\ 1.0 \text{ kN} \end{Bmatrix}$$

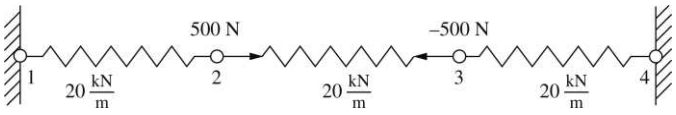
Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0.002 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{Bmatrix} 2.0 \text{ kN} \\ -2.0 \text{ kN} \end{Bmatrix}$$

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2.16



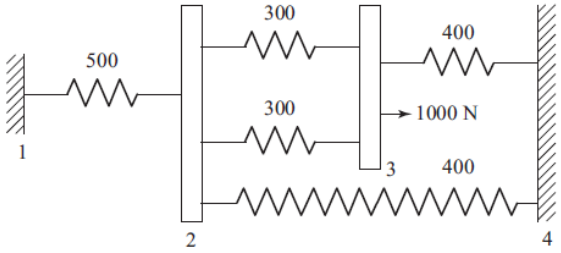
$$\begin{Bmatrix} F_{1x} \\ 500 \\ -500 \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 20 & -200 & 0 & 0 \\ -20 & 20+20 & -20 & 0 \\ 0 & -20 & 20+20 & -20 \\ 0 & 0 & -20 & 20 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 500 \\ -500 \end{Bmatrix} = 10^3 \begin{bmatrix} 40 & -20 \\ -20 & 40 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$u_2 = 8.33 \times 10^{-3}\ \text{m} = 8.33\ \text{mm}$$

$$u_3 = -8.33 \times 10^{-3}\ \text{m} = -8.33\ \text{mm}$$

2.17



$$\begin{Bmatrix} F_{1x} = ? \\ 0 \\ 1000\ \text{N} \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 500 & -500 & 0 & 0 \\ -500 & (400+300)+(500+300) & -300-300 & -400 \\ 0 & -300-300 & (300+300+400) & -400 \\ 0 & -400 & -400 & 400+400 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 = 0 \end{Bmatrix}$$

$$0 = 1500\, u_2 - 600\, u_3$$

$$1000 = -600\, u_2 + 1000\, u_3$$

$$u_3 = \frac{15\cancel{0}\,\cancel{0}}{6\cancel{0}\,\cancel{0}}\, u_2 = 2.5\, u_2$$

$$1000 = -600\, u_2 + 1000\,(2.5\, u_2)$$

$$1000 = 1900\, u_2$$

$$u_2 = \frac{1000}{1900} = \frac{1}{1.9}\ \text{mm} = 0.526\ \text{mm}$$

$$u_3 = 2.5\left(\frac{1}{1.9}\right)\ \text{mm} = 1.316\ \text{mm}$$

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$$F_{1x} = -500\left(\frac{1}{1.9}\right) = -263.16 \text{ N}$$

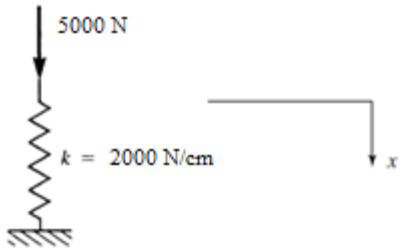
$$F_{4x} = -400\left(\frac{1}{1.9}\right) - 400\left\{2.5\left(\frac{1}{1.9}\right)\right\}$$

$$= -400\left(\frac{1}{1.9} + \frac{2.5}{1.9}\right) = -736.84 \text{ N}$$

$$\Sigma F_x = -263.16 + 1000 - 736.84 = 0$$

2.18

(a)



As in Example 2.4

$$\pi_p = U + \Omega$$

$$U = \frac{1}{2} k x^2, \Omega = -Fx$$

Set up table

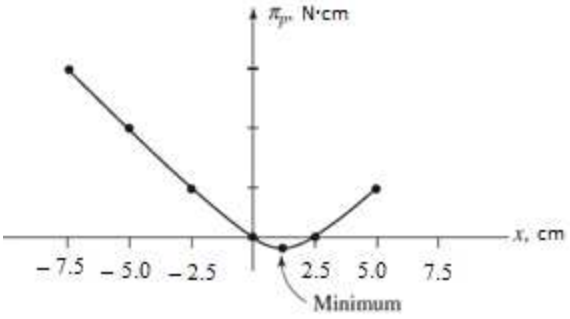
$$\pi_p = \frac{1}{2}(2000)x^2 - 5000x = 1000x^2 - 5000x$$

Deformation x, cm	π_p, N·cm
– 7.5	93,750
– 5.0	50,000
– 2.5	18,750
0.0	0
0.5	6250
2.5	0
7.5	18,750

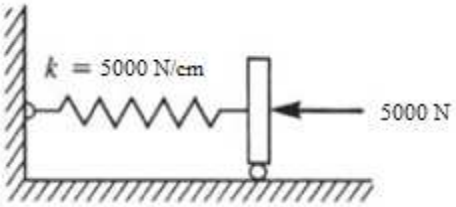
$\frac{\partial \pi_p}{\partial x} = 2000x - 5000 = 0 \Rightarrow x = 2.5$ cm yields minimum π_p as table verifies.

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(b)



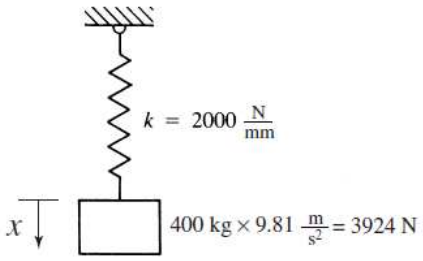
$$\pi_p = \frac{1}{2} kx^2 - F_x = 2500\, x^2 - 5000\, x$$

x, cm	π_p, N·cm
– 3.0	37,5000
– 2.0	20,000
– 1.0	7500
0	0
1.0	– 2500
2.0	0
3.0	7500

$$\frac{\partial \pi_p}{\partial x} = 5000\, x - 5000 = 0$$

$$\Rightarrow \qquad x = 1.0 \text{ in. yields } \pi_p \text{ minimum}$$

(c)



$$\pi_p = \frac{1}{2} (2000)\, x^2 - 3924\, x = 1000\, x^2 - 3924\, x$$

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$$\frac{\partial \pi_p}{\partial x} = 2000\,x - 3924 = 0$$

$$\Rightarrow \quad x = 1.962 \text{ mm yields } \pi_p \text{ minimum}$$

$$\pi_{p\,\min} = \frac{1}{2}\,(2000)\,(1.962)^2 - 3924\,(1.962)$$

$$\Rightarrow \pi_{p\,\min} = -3849.45 \text{ N·mm}$$

(d) $$\pi_p = \frac{1}{2}\,(400)\,x^2 - 981\,x$$

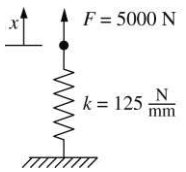
$$\frac{\partial \pi_p}{\partial x} = 400\,x - 981 = 0$$

$$\Rightarrow \quad x = 2.4525 \text{ mm yields } \pi_p \text{ minimum}$$

$$\pi_{p\,\min} = \frac{1}{2}\,(400)\,(2.4525)^2 - 981\,(2.4525)$$

$$\Rightarrow \pi_{p\,\min} = -1202.95 \text{ N·mm}$$

2.19



Now let positive x be upward

$$\pi_p = \frac{1}{2}\,kx^2 - Fx$$

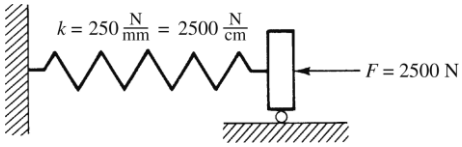
$$\pi_p = \frac{1}{2}\,(125)\,x^2 - 5000\,x$$

$$\pi_p = (62.5)x^2 - 5000\,x$$

$$\frac{\partial \pi_p}{\partial x} = (125)x - 5000 = 0$$

$$\Rightarrow x = 40.0 \text{ mm} \uparrow$$

2.20



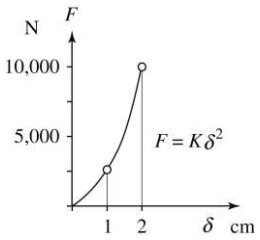
$$F = k\delta^2 \qquad (x = \delta)$$

$$dU = F\,dx$$

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$$U = \int_0^x (kx^2)\, dx$$



$$U = \frac{k x^3}{3}$$

$$\Omega = -Fx$$

$$\pi_p = \frac{1}{3}\, kx^3 - 2500\, x$$

$$\frac{\partial \pi_p}{\partial x} = 0 = kx^2 - 2500$$

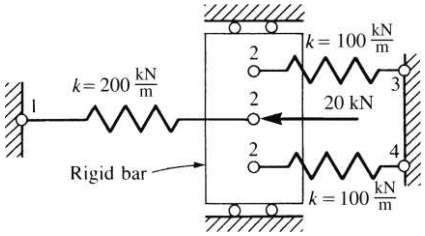
$$0 = 2500\, x^2 - 2500$$

$$\Rightarrow \quad x = 1.0 \text{ cm} \quad \text{(equilibrium value of displacement)}$$

$$\pi_{p\ \min} = \frac{1}{3}\, (2500)\, (1.0)^3\, (2500)\, (7.0)$$

$$\pi_{p\ \min} = -1666.7 \text{ N} \cdot \text{cm}$$

2.21 Solve Problem 2.10 using P.E. approach



$$\pi_p = \sum_{e=1}^{3} \pi_p^{(e)} = \frac{1}{2}\, k_1 (u_2 - u_1)^2 + \frac{1}{2}\, k_2 (u_3 - u_2)^2 + \frac{1}{2}\, k_3 (u_4 - u_2)^2$$

$$- f_{1x}^{(1)}\, u_1 - f_{2x}^{(1)}\, u_2 - f_{2x}^{(2)}\, u_2$$

$$- f_{3x}^{(2)}\, u_3 - f_{2x}^{(3)}\, u_2 - f_{4x}^{(3)}\, u_4$$

$$\frac{\partial \pi_p}{\partial u_1} = -k_1 u_2 + k_1 u_1 - f_{1x}^{(1)} = 0 \tag{1}$$

$$\frac{\partial \pi_p}{\partial u_2} = k_1 u_2 - k_1 u_1 - k_2 u_3 + k_2 u_2 - k_3 u_4$$
$$+ k_3 u_2 - f_{2x}^{(1)} - f_{2x}^{(2)} - f_{2x}^{(3)} = 0 \tag{2}$$

$$\frac{\partial \pi_p}{\partial u_3} = k_2 u_3 - k_2 u_2 - f_{3x}^{(2)} = 0 \tag{3}$$

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$$\frac{\partial \pi_p}{\partial u_4} = k_3 u_4 - k_3 u_2 - f_{4x}^{(3)} = 0 \tag{4}$$

In matrix form (1) through (4) become

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} + f_{2x}^{(2)} + f_{2x}^{(3)} \\ f_{3x}^{(2)} \\ f_{4x}^{(3)} \end{Bmatrix} \tag{5}$$

or using numerical values

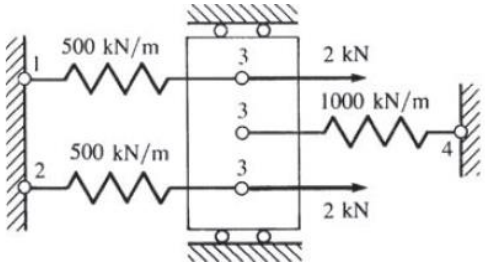
$$10^3 \times \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 400 & -200 & -200 \\ 0 & -200 & 200 & 0 \\ 0 & -200 & 0 & 200 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ -20{,}000 \\ F_{3x} \\ F_{4x} \end{Bmatrix} \tag{6}$$

Solution now follows as in Problem 2.10

Solve 2nd of Equations (6) for $u_2 = -0.05$ m = 5 cm

For reactions and element forces, see solution to Problem 2.10

2.22 Solve Problem 2.15 by P.E. approach



$$\pi_p = \sum_{e=1}^{3} \pi_p^{(e)} = \frac{1}{2} k_1 (u_3 - u_1)^2 + \frac{1}{2} k_2 (u_3 - u_2)^2$$

$$+ \frac{1}{2} k_3 (u_4 - u_3)^2 - f_{1x}^{(1)} u_1$$

$$- f_{3x}^{(1)} u_3 - f_{2x}^{(2)} u_2 - f_{3x}^{(2)} u_3$$

$$- f_{3x}^{(3)} u_3 - f_{3x}^{(4)} u_4$$

$$\frac{\partial \pi_p}{\partial u_1} = 0 = -k_1 u_3 + k_1 u_1 - f_{1x}^{(1)}$$

$$\frac{\partial \pi_p}{\partial u_2} = 0 = -k_2 u_3 + k_2 u_2 - f_{2x}^{(2)}$$

$$\frac{\partial \pi_p}{\partial u_3} = 0 = k_1 u_3 + k_2 u_3 - k_2 u_2 - k_3 u_4 + k_3 u_3 - f_{3x}^{(2)} - f_{3x}^{(3)} - f_{3x}^{(1)} - k_1 u_1$$

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$$\frac{\partial \pi_p}{\partial u_4} = 0 = k_3 u_4 - k_3 u_3 - f_{3x}^{(4)}$$

In matrix form

$$\begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_2 & -k_2 & 0 \\ -k_1 & -k_2 & k_1+k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} = 4 \text{ kN} \\ F_{4x} \end{Bmatrix}$$

For rest of solution, see solution of Problem 2.15.

2.23

$$I = a_1 + a_2 x$$

$$I(0) = a_1 = I_1$$

$$I(L) = a_1 + a_2 L = I_2$$

$$a_2 = \frac{I_2 - I_1}{L}$$

$$\therefore \qquad I = I_1 + \frac{I_2 - I_1}{L} x$$

Now $V = IR$

$$V = -V_1 = R(I_2 - I_1)$$

$$V = V_2 = R(I_2 - I_1)$$

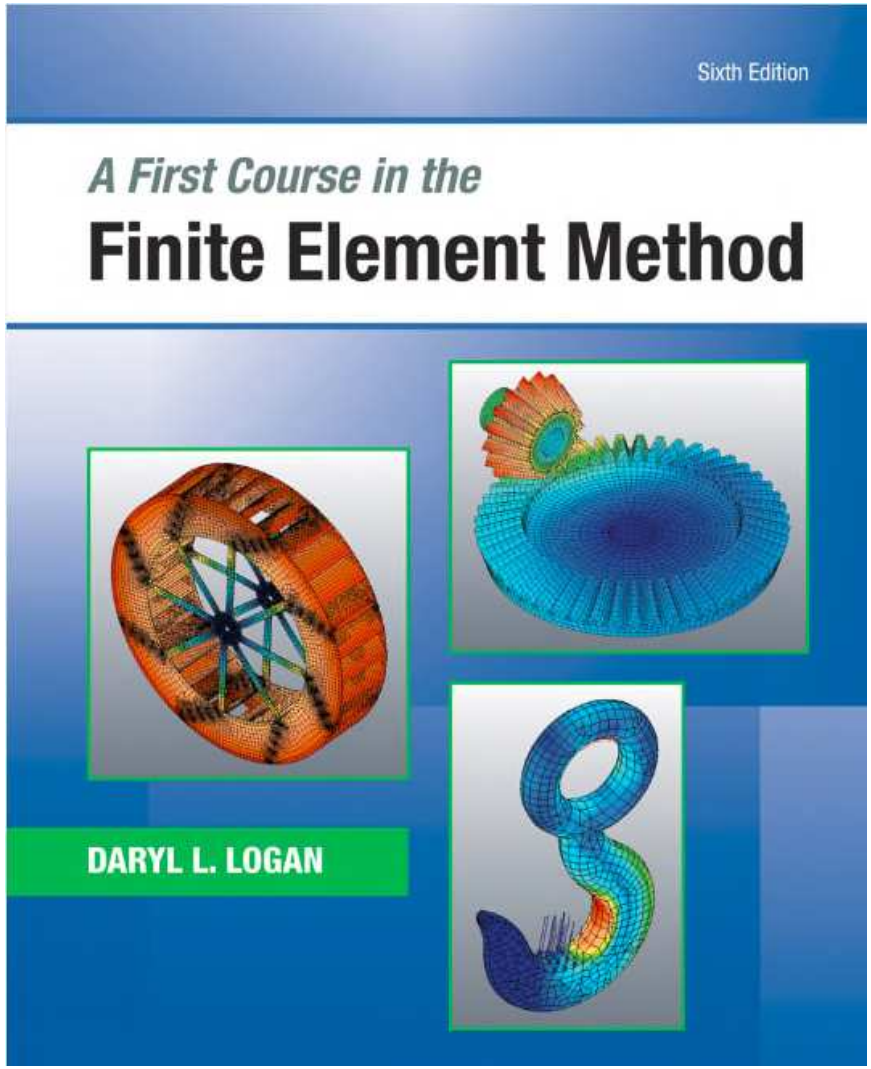
$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = R \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix}$$

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Chapter 2

Introduction to the Stiffness (Displacement) Method

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Learning Objectives

- To define the stiffness matrix
- To derive the stiffness matrix for a spring element
- To demonstrate how to assemble stiffness matrices into a global stiffness matrix
- To illustrate the concept of direct stiffness method to obtain the global stiffness matrix and solve a spring assemblage problem
- To describe and apply the different kinds of boundary conditions relevant for spring assemblages
- To show how the potential energy approach can be used to both derive the stiffness matrix for a spring and solve a spring assemblage problem

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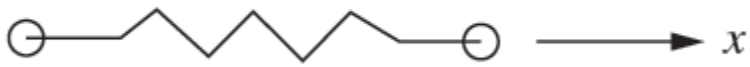
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Definition of the Stiffness Matrix

- For an element, a stiffness matrix [k] is a matrix such that:

$$\{f\} = [k]\{d\}$$

Where [k] relates nodal displacements {d} to nodal forces {f} of a single element, such as to the single spring element below



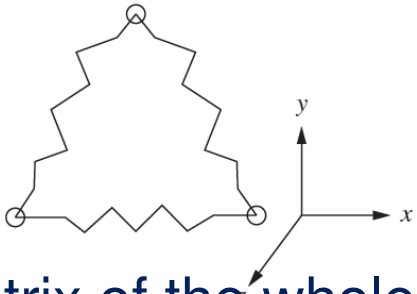
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Definition of the Stiffness Matrix

- For a structure comprising of a series of elements such as the three-spring assemblage shown below:



- The stiffness matrix of the whole spring assemblage [K] relates global-coordinate nodal displacements {d} to global forces {F} by the relation:

$$\{F\} = [K]\{d\}$$

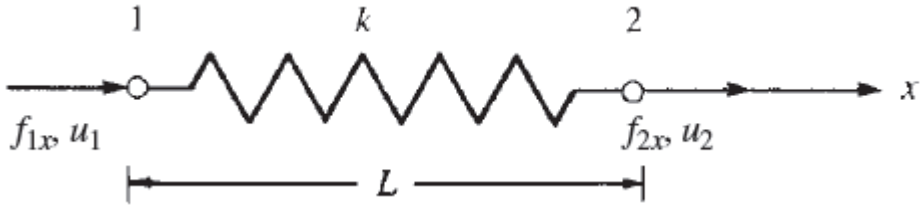
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Derivation of the Stiffness Matrix for a Spring Element

- Consider the following linear spring element:



- Points 1 and 2 are reference points called nodes
- f_{1x} and f_{2x} are the local nodal forces on the x-axis
- μ_1 and μ_2 are the local nodal displacements
- k is the spring constant or stiffness of the spring
- L is the distance between the nodes

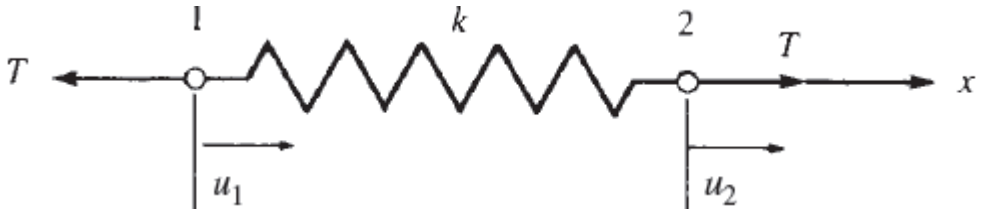
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Derivation of the Stiffness Matrix for a Spring Element

- We have selected our element type and now need to define the deformation relationships



- For the spring subject to tensile forces at each node:

  $\delta = \mu_2 - \mu_1$ & $T = k\delta$

  Where δ is the total deformation and T is the tensile force
- Combine to obtain: $T = k(\mu_2 - \mu_1)$

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Derivation of the Stiffness Matrix for a Spring Element

- Performing a basic force balance yields:

$$f_{1x} = -T \qquad f_{2x} = T$$

- Combining these force eqs with the previous eqs:

$$f_{1x} = k(u_1 - u_2)$$

$$f_{2x} = k(u_2 - u_1)$$

- Express in matrix form:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

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Derivation of the Stiffness Matrix for a Spring Element

- The stiffness matrix for a linear element is derived as:

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

- Here [k] is called the local stiffness matrix for the element.
- Observe that this matrix is symmetric, is square, and is singular.
- This was the basic process of deriving the stiffness matrix for any element.

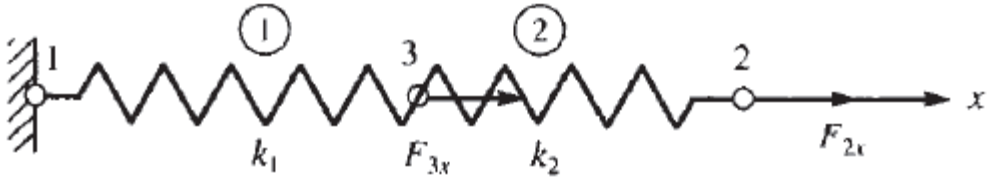
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Establishing the Global Stiffness Matrix for a Spring Assemblage

- Consider the two-spring assemblage:



- Node 1 is fixed and axial forces are applied at nodes 3 and 2.
- The x-axis is the global axis of the assemblage.

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Establishing the Global Stiffness Matrix for a Spring Assemblage

- For element 1:

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{3x}^{(1)} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_3^{(1)} \end{Bmatrix}$$

- For element 2:

$$\begin{Bmatrix} f_{3x}^{(2)} \\ f_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_3^{(2)} \\ u_2^{(2)} \end{Bmatrix}$$

- Elements 1 and 2 must remain connected at common node 3. The is called the continuity or compatibility requirement given by:

$$u_3^{(1)} = u_3^{(2)} = u_3$$

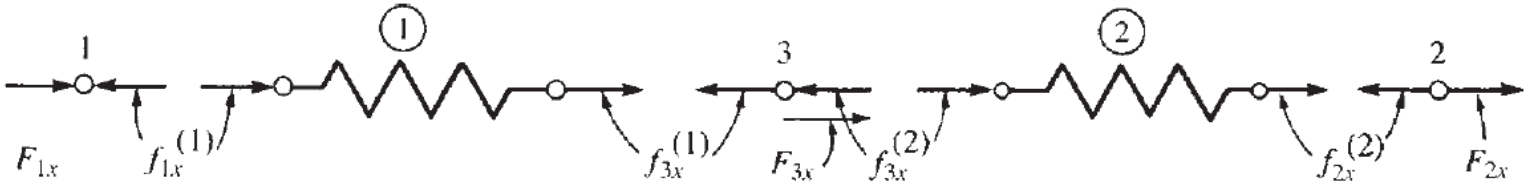
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Establishing the Global Stiffness Matrix for a Spring Assemblage

- From the Free-body diagram of the assemblage:



- We can write the equilibrium nodal equations:

$$F_{3x} = f_{3x}^{(1)} + f_{3x}^{(2)} \qquad F_{2x} = f_{2x}^{(2)} \qquad F_{1x} = f_{1x}^{(1)}$$

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Establishing the Global Stiffness Matrix for a Spring Assemblage

- Combining the nodal equilibrium equations with the elemental force/displacement/stiffness relations we obtain the global relationship:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

- Which takes the form: {F} = [K]{d}
- {F} is the global nodal force matrix
- {d} is the global nodal displacement matrix
- [K] is the total or global or system stiffness matrix

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Direct Stiffness Method

- Reliable method of directly assembling individual element stiffness matrices to form the total structure stiffness matrix and the total set of stiffness equations
- Individual element stiffness matrices are superimposed to obtain the global stiffness matrix.
- To superimpose the element matrices, they must be expanded to the order (size) of the total structure stiffness matrix.

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Boundary Conditions

- We must specify boundary (or support) conditions for structure models or [K] will be singular.
- This means that the structural system is unstable.
- Without specifying proper kinematic constraints or support conditions, the structure will be free to move as a rigid body and not resist any applied loads.
- In general, the number of boundary conditions necessary is equal to the number of possible rigid body modes.

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Boundary Conditions

- Homogeneous boundary conditions
  - Most common type
  - Occur at locations completely prevented from moving
  - Zero degrees of freedom
- Nonhomogeneous boundary conditions
  - Occur where finite nonzero values of displacements are specified
  - Nonzero degree of freedom
  - i.e. the settlement of a support

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Homogenous Boundary Conditions

- Where is the homogenous boundary condition for the spring assemblage?
- It is at the location which is fixed, Node 1
- Because Node 1 is fixed $\mu_1 = 0$
- The system relation can be written as:

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

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Homogenous Boundary Conditions

- For all homogenous boundary conditions, we can delete the row and columns corresponding to the zero-displacement degrees of freedom.
- This makes solving for the unknown displacements possible.
- Appendix B.4 presents a practical, computer-assisted scheme for solving systems of simultaneous equations.

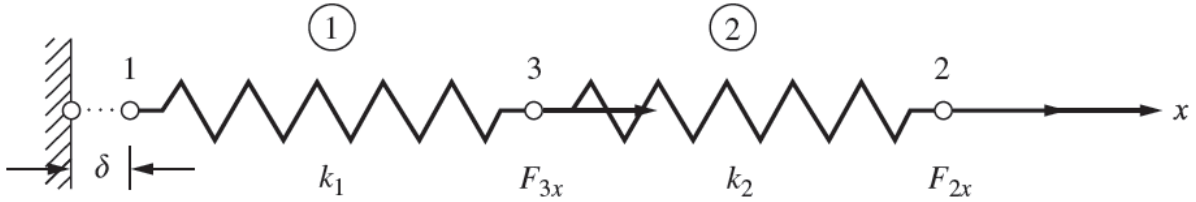
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Nonhomogeneous Boundary Conditions

- Consider the case where there is a known displacement, δ, at Node 1



- Let $\mu_1 = \delta$

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} \delta \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

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Nonhomogeneous Boundary Conditions

- By considering only the second and third force equations we can arrive at the equation:

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{2x} \\ k_1\delta + F_{3x} \end{Bmatrix}$$

- It can be seen that for nonhomogeneous boundary conditions we cannot initially delete row 1 and column 1 like was done for homogenous boundary conditions.

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Nonhomogeneous Boundary Conditions

- In general for nonhomogeneous boundary conditions, we must transform the terms associated with the known displacements to the force matrix before solving for the unknown nodal displacements.

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Minimum Potential Energy Approach

- Alternative method often used to derive the element equations and stiffness matrix.
- More adaptable to the determination of element equations for complicated elements such as:
  - Plane stress/strain element
  - Axisymmetric stress element
  - Plate bending element
  - Three-dimensional solid stress element

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Minimum Potential Energy Approach

- Principle of minimum potential energy is only applicable to elastic materials.
- Categorized as a “variational method” of FEM
- Use the potential energy approach to derive the spring element equations as we did earlier with the direct method.

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Total Potential Energy

- Defined as the sum of the internal strain energy, U, and the potential energy of the external forces, Ω

$$\pi_p = U + \Omega$$

- Strain energy is the capacity of internal forces to do work through deformations in the structure.
- The potential energy of external forces is the capacity of forces such as body forces, surface traction forces, or applied nodal forces to do work through deformation of the structure.

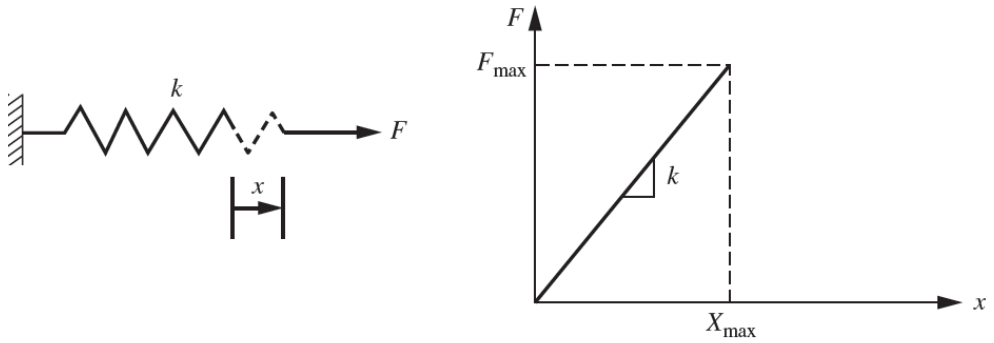
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Concept of External Work

- A force is applied to a spring and the force-deformation curve is given.
- The external work is given by the area under the force-deformation curve where the slope is equal to the spring constant k



(a) (b)

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External Work and Internal Strain Energy

- From basic mechanics principles the external work is expressed as:

$$W_e = \int F \cdot dx = \int_0^{x_{\max}} F_{\max}\left(\frac{x}{x_{\max}}\right) dx = F_{\max} x_{\max}/2$$

- From conservation of mechanical energy principle external work is expressed as:

$$W_e = U = F_{\max} x_{\max}/2$$

- For when the external work is transformed into the internal strain energy of the spring

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Total Potential Energy of Spring

- The strain energy can be expressed as:

$$U = kx_{\max}^2/2$$

- The potential energy of the external force can be expressed as:

$$\Omega = -F_{\max}x_{\max}$$

- Therefore, the total potential energy of a spring is:

$$\pi_p = \frac{1}{2}kx_{\max}^2 - F_{\max}x_{\max}$$

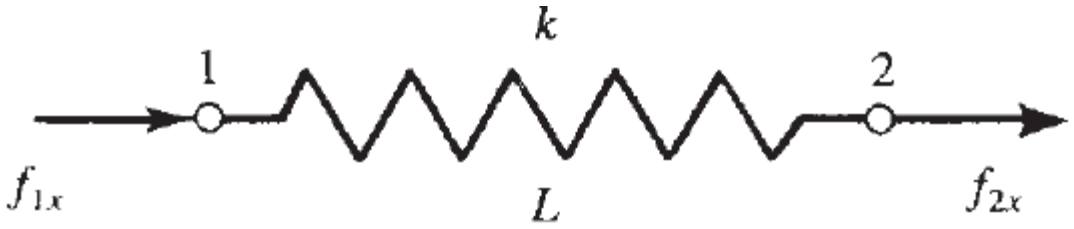
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Potential Energy Approach to Derive Spring Element Eqs.

- Consider the linear spring subject to nodal forces:



- The total potential energy is:

$$\pi_p = \frac{1}{2}k(u_2 - u_1)^2 - f_{1x}u_1 - f_{2x}u_2$$

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Potential Energy Approach to Derive Spring Element Eqs.

- To minimize the total potential energy the partial derivatives of π_p with respect to each nodal displacement must be taken:

$$\frac{\partial \pi_p}{\partial u_1} = \frac{1}{2}k(-2u_2 + 2u_1) - f_{1x} = 0$$

$$\frac{\partial \pi_p}{\partial u_2} = \frac{1}{2}k(2u_2 - 2u_1) - f_{2x} = 0$$

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Potential Energy Approach to Derive Spring Element Eqs.

- Simplify to:

$$k(-u_2 + u_1) = f_{1x}$$
$$k(u_2 - u_1) = f_{2x}$$

- In matrix form:

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix}$$

- The results are identical to the direct method

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Summary

- Defined the stiffness matrix
- Derived the stiffness matrix for a spring element
- Established the global stiffness matrix for a spring assemblage
- Discussed boundary conditions (homogenous & nonhomogeneous)
- Introduced the potential energy approach
- Reviewed minimum potential energy, external work, and strain energy
- Derived the spring element equations using the potential energy approach

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