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CHAPTER 2

Maxwell's equation and Electromagnetic Waves

2.1 Consider a vector field $\mathbf{V}(\mathbf{r}) = 4x\mathbf{x}_0 + 5y\mathbf{y}_0 + 6z\mathbf{z}_0$ and a closed, cubical S surface with side length L and one corner at the origin, lying in the first octant. Evaluate the integral $\oint_S \mathbf{V} \cdot d\mathbf{A}$ by first carrying out the dot product and integral on each of the six faces of the cube, and adding them up. Check your answer by using the divergence theorem, which you are likely able to do in your head.

Solution
Make a table to explicitly evaluate the surface integral:

Surface	$\int \mathbf{V} \cdot d\mathbf{A}$
xy @ $z = 0$	0
xy @ $z = L$	$6L \cdot L^2$
yz @ $x = 0$	0
yz @ $x = L$	$4L \cdot L^2$
zx @ $y = 0$	0
zx @ $y = L$	$5L \cdot L^2$
Sum	$15L^3$

On the other hand, $\int \boldsymbol{\nabla} \cdot \mathbf{V} d\tau = 15 \int d\tau = 15L^3$.

2.2 Show that the integral form of Coulomb's law can be derived from Gauss's law. First, argue why rotational symmetry implies that the electric field from a point charge q has to be isotropic in all directions, and can only depend on the distance

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r from the charge. Next, use this to choose an appropriate "Gaussian surface" S so that the integral in Equation (2.4a) is simple to evaluate. Finally, use Equation (2.5) to show that the force F on another charge q' is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{r^2}$$

Solution
It is obvious that the field can only depend on r because there is no preferred direction. Similarly, it can only be radially outward (or inward), so choose a Gaussian surface that is a sphere of radius r centered at the origin. The magnitude E of the electric field is given by (2.4a) as

$$E \cdot 4\pi r^2 = \frac{q}{\epsilon_0} \qquad \text{so} \qquad E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

and the force on a charge q' is just $q'E$.

2.3 A "parallel plate capacitor" is made from two plane conducting sheets, each with area A, separated by a distance d. The plates carry equal but opposite charges $\pm Q$, uniformly distributed over their surface, and this creates a potential difference V between them. Infer the (constant) electric field between the plates, and use Gauss's law to show that $Q = CV$, where C depends only A and d (and ϵ_0).

Solution

The surface charge density is $\sigma = Q/A$, so a "pillbox" Gaussian surface with one flat surface inside the metal plate (where the field is zero) and the other flat surface in the gap, gives $E = \sigma/\epsilon_0$. The potential difference for this (constant) electric field is just $V = Ed = \sigma d/\epsilon_0 = Qd/\epsilon_0 A$ so that $C = Q/V = \epsilon_0 A/d$.

2.4 Use the concept of a parallel plate capacitor to find the energy density in an electric field. Charge is added in small increments dQ' to an initially uncharged capacitor giving a potential difference V'. Each increment changes the stored energy by $V'dQ' = (Q'/C)dQ'$ where C is the capacitance. (See Problem 2.3.) Integrate to find the total energy when charge Q is stored in the capacitor. Divide by the volume of the capacitor to find the electric field energy density

$$u_E = \frac{1}{2}\epsilon_0 E^2$$

where E is the electric field inside the capacitor.

Solution
Just do as the problem statement tells you to do:

$$u_e = \frac{1}{Ad}U = \frac{1}{Ad}\int_0^Q \frac{1}{C}Q'dQ' = \frac{1}{Ad}\frac{Q^2}{2C} = \frac{1}{Ad}\frac{(\epsilon_0 EA)^2}{2\epsilon_0 A/d} = \frac{1}{2}\epsilon_0 E^2$$

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2.5 Calculate the magnetic field at a distance r from an infinitely long straight wire which carries a current I. First, using Gauss's law for magnetism, explain why the field must be tangential to a circle of radius r, centered on the wire and lying in a plan perpendicular to the wire. Then use Ampére's law to show that the magnitude of the magnetic field is

$$B = \frac{\mu_0 I}{2\pi r}$$

Solution
The problem has cylindrical symmetry, but there is a handedness set by the direction of the current. By Gauss' Law for magnetism, there can be no radial component, as a cylindrical Gaussian surface can pass no no flux. With a circular loop at radius r, the line integral of Ampere's Law (2.4d) is just $B \cdot 2\pi r$ for an azimuthal field $\mathbf{B}$, hence $B = \mu_0 I/2\pi r$.

2.6 A long cylindrical coil of wire is called a *solenoid* and can be used to store a magnetic field. If the coil is infinitely long, there is a uniform magnetic field in the axial direction inside the coil, and no field outside the coil. Use an "Amperian Loop" that is a rectangle enclosing some length of the coil, with one leg inside and one leg outside, to show that the magnetic field is

$$B = \mu_0 I n$$

where I is the current in the wire and there are n turns per unit length in the coil.

Solution
The current enclosed in the rectangular loop is $nI\ell$ where ℓ is the length of the loop. There is no field outside the solenoid, and the field inside is parallel to the axis, so the line integral just gives $B\ell$, hence $B = \mu_0 nI$.

2.7 Find the vector potential $\mathbf{A}(\mathbf{r})$ which gives the magnetic field for the long straight wire in Problem 2.5. It is easiest to let the wire lie along the z-axis and express your result in terms of $r = (x^2 + y^2)^{1/2}$, and to carry out the calculation in cylindrical polar coordinates (r, θ, z).

Solution
$\mathbf{B} = B\hat{\theta} = \boldsymbol{\nabla} \times \mathbf{A} = -(\partial A_z/\partial r)\hat{\theta}$, so $\partial A_z/\partial r = -\mu_0 I/2\pi r$ and $\mathbf{A} = -(\mu_0 I/2\pi) \log r\, \hat{\mathbf{z}}$.

2.8 Follow this guide to convince yourself that the second term on the right in (2.4d) is needed for the whole equation to make sense. First, imagine a long straight current-carrying wire, with associated magnetic field given in Problem 2.5. Now "cut" the wire, and insert a very thin capacitor, with plates perpendicular to the direction of the wire. Current continues to flow through the wire while the capacitor charges up, but no current flows between the capacitor plates, so it would seem there should be no magnetic field there. But that doesn't make sense: how could the magnetic field just stop at the capacitor? Intuitively, you expect it to be continuous right through it.

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Finally, show that the second term, called a *displacement current*, in fact gives you the same B inside the capacitor.

Solution The capacitor plates are circular with area $A = \pi r^2$. The electric field flux through this area is $A\sigma/\epsilon_0 = Q/\epsilon_0$, since $\sigma = Q/A$. The left side of (2.4d) must equal $\mu_0 I$, where $I = dQ/dt$, so

$$\mu_0 I = K\frac{d}{dt}\left[\frac{Q}{\epsilon_0}\right] = K\frac{I}{\epsilon_0}$$

which implies that $K = \epsilon_0\mu_0$, establishing (2.4d).

2.9 A "$1/r^2$" vector field, such as the electric field from a point charge or the gravitational field from a point mass, takes the form

$$\mathbf{V}(\mathbf{r}) = \frac{k}{r^2}\mathbf{r}_0 = \frac{k}{r^3}\mathbf{r}$$

Show by an explicit calculation in Cartesian coordinate coordinates, that $\nabla\cdot\mathbf{V} = 0$ everywhere, except at the origin. Then, using a spherical surface about the origin, show that $\oint \mathbf{V}\cdot d\mathbf{A} = 4\pi k$. Hence argue that the charge density for a point charge q located at the origin is $q\delta^3(\mathbf{r}) = q\delta(x)\delta(y)\delta(z)$, where $\delta(x)$ is a Dirac $\delta(x)$ function as defined in Chapter 1.

Solution
The calculation is straightforward, although a bit tedious:

$$\begin{aligned}\nabla\cdot\frac{k}{r^3}\mathbf{r} &= k\left[\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}}\right]\\ &= \frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5} = \frac{3r^2 - 3(x^2+y^z+z^2)}{r^5} = 0\end{aligned}$$

for $r \neq 0$. For a sphere of radius r about the origin, $\oint \mathbf{V}\cdot d\mathbf{A} = (k/r^2)\cdot 4\pi r^2 = 4\pi k$. So, by Gauss' Theorem for this spherical volume, $\int \nabla\cdot\mathbf{V} = 4\pi k$, but since $\nabla\cdot\mathbf{V} = 0$ everywhere except the origin, consider a small cube around the origin, and it is clear that $\nabla\cdot\mathbf{V}$ satisfies the properties of the 3D δ-function, i.e. $\nabla\cdot\mathbf{V} = 4\pi k\delta(x)\delta(y)\delta(z) = 4\pi k\delta^{(3)}(\mathbf{r})$. For Coulomb's Law, $\mathbf{V} = \mathbf{E}$ and $k = q/4\pi\epsilon_0$, so Gauss' Law takes the form $\nabla\cdot\mathbf{E} = q\delta^{(3)}(\mathbf{r})/\epsilon_0$. Comparing to (2.19a), this implies $\rho(\mathbf{r}) = q\delta^{(3)}(\mathbf{r})$ for a point charge q.

2.10 A "$1/r$" vector field, such as the magnetic field from an infinitely long current carrying wire, takes the form

$$\mathbf{V}(\mathbf{r}) = \frac{k}{r}\phi_0 = \frac{k}{r^z}[-y\mathbf{x}_0 + x\mathbf{y}_0]$$

Show by an explicit calculation in Cartesian coordinate coordinates, that $\nabla\times\mathbf{V} = 0$ everywhere, except at the origin. Then, using a circular curve about the z-axis, show that $\oint \mathbf{V}\cdot d\mathbf{l} = 2\pi k$. Hence argue that the current density for an infinitely long current carrying wire of zero thickness located along the z-axis is $I\delta(x)\delta(y)$.

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Solution
The calculation is straightforward, but tedious. Also, note here that $r^2 = x^2 + y^2$.

$$\begin{aligned}\boldsymbol{\nabla}\times\frac{k}{r^2}(-y\mathbf{x}_0+x\mathbf{y}_0) &= k\left[\frac{1}{r^2}\frac{\partial x}{\partial x}+x\frac{\partial}{\partial x}\frac{1}{x^2+y^2}+\frac{1}{r^2}\frac{\partial y}{\partial y}+y\frac{\partial}{\partial y}\frac{1}{x^2+y^2}\right]\mathbf{z}_0\\ &= \frac{k}{r^2}\left[1-\frac{2x^2}{r^2}+1-\frac{2y^2}{r^2}\right]\mathbf{z}_0=\frac{k}{r^4}\left[2-2x^2 2y^2\right]\mathbf{z}_0=0\end{aligned}$$

except for $r\neq 0$. For the circular loop, $\oint \mathbf{V}\cdot d\mathbf{l} = (k/r)(2\pi r) = 2\pi k$. Therefore, using the magnetic field $\mathbf{B}$ from a line current, we get $k = \mu_0 I/2\pi$. Applying Stokes' Theorem along with (2.19d) implies that $\mathbf{j} = I\delta^{(2)}(\mathbf{r})\mathbf{z}_0$.

2.11 Derive the wave equation for the magnetic field $\mathbf{B}$ from Maxwell's equations.

Solution
Start with Equations (2.19), and take the curl of (2.19b) with $\mathbf{j} = 0$:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{B}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{B})-\boldsymbol{\nabla}^2\mathbf{B} = -\boldsymbol{\nabla}^2\mathbf{B} = \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{B}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}$$

since $\mu_0\epsilon_0 = 1/c^2$, and using (2.32) with (2.19c) and (2.19b). This gives

$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}-\boldsymbol{\nabla}^2\mathbf{B}=0$$

2.12 Show that (2.39) is a solution to the wave Equation (2.34). It is easier to do if you express the Laplacian in spherical coordinates, but you may find it more satisfying to work it through in Cartesian coordinate coordinates, remembering that $r = (x^2+y^2+z^2)^{1/2}$.

Solution
There is a typo in the problem. It should ask to show that (2.41) is a solution, not (2.39). That is, taking time derivatives and with $\omega = kc$, we need to show that

$$\boldsymbol{\nabla}^2\left[\frac{f(kr-\omega t)}{r}g(\theta,\phi)\right] = k^2\frac{r^2 f''(kr-\omega t)}{r}g(\theta,\phi)$$

The Laplacian in spherical coordinates is easy enough to find, for example (6.19). So,

$$g\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{f}{r}\right)+\frac{f}{r^3}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\phi^2}\right] = k^2\frac{f''}{r}g$$

Now work on the first term on the left hand side of the above equation:

$$g\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{f}{r}\right) = g\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}-f\right) = g\frac{1}{r^2}\left(\frac{\partial f}{\partial r}+r\frac{\partial^2 f}{\partial r^2}-\frac{\partial f}{\partial r}\right) = g\frac{1}{r}k^2 f''$$

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That is, this cancels with the right side of the wave equation, leaving

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\phi^2}=0$$

which is a differential equation that can be solved for $g(\theta,\phi)$.

2.13 Find the electrostatic potential $V(\mathbf{r})$ at a distance r from a point charge q, with the integration constant defined so that $V(\mathbf{r})\to 0$ as $r\to\infty$. You can do this either by directly integrating $\int_C \mathbf{E}\cdot d\mathbf{l}$ along some path C, or by simply guessing the function $V(\mathbf{r})$ which satisfies (2.20).

Solution
Let the curve C be the straight line starting at $\mathbf{r}$ and extending radially to ∞. That is, $d\mathbf{l}=d\mathbf{r}$ and with $\mathbf{E}=(1/4\pi\epsilon_0)q\mathbf{r}/r^3$, we have

$$V(\mathbf{r})=-\int_r^\infty \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}r dr=-\frac{q}{4\pi\epsilon_0}\int_r^\infty\frac{1}{r^2}dr=\frac{q}{4\pi\epsilon_0}\frac{1}{r}\Big|_r^\infty=\frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

Going the other way, and using Cartesian coordinates, we need to calculate

$$-\boldsymbol{\nabla}V(\mathbf{r})=-\frac{q}{4\pi\epsilon_0}\left[\mathbf{x}_0\frac{\partial}{\partial x}+\mathbf{y}_0\frac{\partial}{\partial y}+\mathbf{z}_0\frac{\partial}{\partial z}\right]\frac{1}{(x^2+y^2+z^2)^{1/2}}=\frac{q}{4\pi\epsilon_0}\frac{x\mathbf{x}_0+y\mathbf{y}_0+z\mathbf{z}_0}{(x^2+y^2+z^2)^{3/2}}=\frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$$

2.14 An "electric dipole" is formed by two point charges $\pm q$ separated by a distance d. Form such a system by putting the two charges on the z-axis at $z=\pm d/2$ and calculate the electric field $\mathbf{E}(\mathbf{r})$. Show that for distances $r\gg d$, the electric field can be written in terms of the *electric dipole moment* vector $\mathbf{p}=qd\mathbf{z}_0$ instead of explicitly on either q or d.

Solution
Just go ahead and add the electric fields of the two charges:

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{r}-(d/2)\hat{\mathbf{z}}_0}{|\mathbf{r}-(d/2)\hat{\mathbf{z}}_0|^3}-\frac{\mathbf{r}+(d/2)\hat{\mathbf{z}}_0}{|\mathbf{r}+(d/2)\hat{\mathbf{z}}_0|^3}\right]\\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^3}\left[\frac{\mathbf{r}-(d/2)\hat{\mathbf{z}}_0}{|\hat{\mathbf{r}}-(d/2r)\hat{\mathbf{z}}_0|^3}-\frac{\mathbf{r}+(d/2)\hat{\mathbf{z}}_0}{|\hat{\mathbf{r}}+(d/2r)\hat{\mathbf{z}}_0|^3}\right]\end{aligned}$$

For $r\gg d$, the denominators can be expanded using

$$\begin{aligned}|\hat{\mathbf{r}}\mp(d/2r)\hat{\mathbf{z}}_0| &= [(\hat{\mathbf{r}}\mp(d/2r)\hat{\mathbf{z}}_0)\cdot(\hat{\mathbf{r}}\mp(d/2r)\hat{\mathbf{z}}_0)]^{1/2}\\ &= \left[(1+d^2/4r^2)\mp(d/r)\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}_0\right]^{1/2}\\ &\approx 1\mp\frac{d}{2r}\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}_0\end{aligned}$$

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Therefore

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^3}\left[\left(\mathbf{r}-\frac{d}{2}\hat{\mathbf{z}}_0\right)\left(1+\frac{3d}{2r}\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}_0\right)-\left(\mathbf{r}+\frac{d}{2}\hat{\mathbf{z}}_0\right)\left(1-\frac{3d}{2r}\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}_0\right)\right]\\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^3}\left[\frac{3d}{r}\mathbf{r}(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}_0)-d\hat{\mathbf{z}}_0+O\left(\frac{d^2}{r^2}\right)\right]\\ &= \frac{1}{4\pi\epsilon_0}\frac{qd}{r^3}\left[3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}_0)-\hat{\mathbf{z}}_0\right]=\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p})-\mathbf{p}\right]\end{aligned}$$

2.15 Find the electrostatic potential $V(\mathbf{r})$ at a distance r from the center of the electric dipole in Problem 2.14. Express your answer in terms of the electric dipole moment $\mathbf{p}$.

Solution

We need to find the $V(\mathbf{r})$ that satisfies $\mathbf{E}=-\boldsymbol{\nabla}V$. Working in spherical polar coordinates with $\hat{\mathbf{r}}\cdot\mathbf{p}=p\cos\theta$ and $\hat{\mathbf{z}}_0=\hat{\mathbf{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta$, this becomes

$$\frac{\partial V}{\partial r}\hat{\mathbf{r}}+\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}}=\frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left[(\hat{\mathbf{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta)-3\hat{\mathbf{r}}\cos\theta\right]=-\frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left[2\hat{\mathbf{r}}\cos\theta+\hat{\boldsymbol{\theta}}\sin\theta\right]$$

The correct form for $V(\mathbf{r})$ is now apparent, namely

$$V(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta=\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

Of course, one can also just add the electric potential of the two charges:

$$\begin{aligned}V(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{r}-(d/2)\hat{\mathbf{z}}_0|}-\frac{1}{|\mathbf{r}+(d/2)\hat{\mathbf{z}}_0|}\right]\\ &= \frac{q}{4\pi\epsilon_0}\left[1+\frac{d}{2r}\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}_0-\left(1-\frac{d}{2r}\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}_0\right)\right]\\ &= \frac{1}{4\pi\epsilon_0}\frac{qd}{r}\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}_0=\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\end{aligned}$$

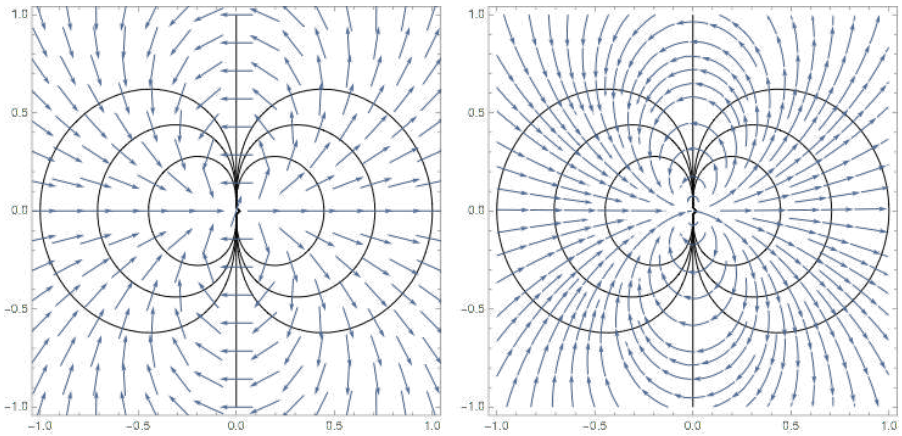
2.16 Draw the electric field lines for and electric dipole with dipole moment $\mathbf{p}=p\mathbf{z}_0$. Superimpose contour lines of electric potential on top of the field lines. This is best done by choosing some specific values and using a program such as MATHEMATICA.

Solution

Making dipole field plots in MATHEMATICA is relatively straightforward, in fact, a module in their documentation shows how to do this in three dimensions. The plots below were made by writing the potential in polar coordinates, calculating the field from the gradient, and using the TransformedField function to transform to Cartesian coordinates before generating the plot. (The dipole lies in the x-direction.) In both cases, ContourPlot is used to plot the potential, specifying the contour line values. The left uses VectorPlot for the field. The right uses StreamPlot, which

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requires less adjustment of options, but which plots only enough arrows so that the density across the plot is more or less constant.

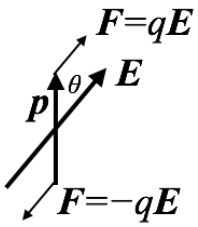


2.17 An electric dipole $\mathbf{p} = qd\mathbf{z}_0$ sits in a uniform, external electric field $\mathbf{E}$. By considering the torque on the dipole, from the interaction of the charges $\pm q$ with $\mathbf{E}$, calculate the work needed to rotate the dipole through an angle θ, about an axis perpendicular to the plane formed by $\mathbf{p}$ and $\mathbf{E}$. Thereby show that the electrostatic potential energy of the dipole in the external field is $U_E = -\mathbf{p}\cdot\mathbf{E}$.

Solution

See the figure. The torque on the dipole is $\tau = 2F(d/2)\sin\theta = qdE\sin\theta$, so the work needed to rotate it to this position is

$$U_E = \int_0^\theta \tau d\theta = -qdE\cos\theta = -\mathbf{p}\cdot\mathbf{E}$$



2.18 A "magnetic dipole" is formed by a planar loop of wire with area A and carrying current I. Using an approach similar to that in Problem 2.17, show that the magnetic potential energy of a magnetic dipole in an external magnetic field $\mathbf{B}$ is $U_B = -\boldsymbol{\mu}\cdot\mathbf{B}$. Here, the magnetic moment $\boldsymbol{\mu}$ has magnitude IA and direction perpendicular to the plane of the loop. You can do this rather easily if you model the loop as a square of side length L, in which case $A = L^2$, then use an argument similar to that used to prove Stokes' theorem to explain why deriving this for a square is equivalent to solving it for any planar loop.

Solution

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This is essentially the same as Problem 17. Orient the square loop so that the normal to the plane and the magnetic field form a plane that is parallel to two sides of the square. Then the force on those two sides just point into, or out of, the loop, with no torque or net force. The force on the other two sides, however, are each ILB with a lever arm $L/2$, so the torque on the loop is $2\times(ILB)(L/2)\sin\theta = (IL^2)B\sin\theta$, where θ is the angle between the loop normal and the magnetic field. Integrating to find the work gives $U_B = -\boldsymbol{\mu}\cdot\mathbf{B}$.

The "Stokes' Theorem" argument just says that we can build up any loop out of a large number of tiny square loops, and the torques add up, as does the area of the big loop.

2.19 Given a scalar function $\chi(\mathbf{r},t)$, show explicitly that the transformation $\mathbf{A}\rightarrow\mathbf{A}+\boldsymbol{\nabla}\chi$ has no effect on the magnetic field. Then find the condition that needs to be satisfied by χ so that the additional transformation $V\rightarrow V+\chi$ has no effect on the electric field.

Solution
For $\mathbf{A}\rightarrow\mathbf{A}'=\mathbf{A}+\boldsymbol{\nabla}\chi$, we have, using (2.24),

$$\mathbf{B}'=\boldsymbol{\nabla}\times\mathbf{A}'=\boldsymbol{\nabla}\times\mathbf{A}+\boldsymbol{\nabla}\times\boldsymbol{\nabla}\chi=\boldsymbol{\nabla}\times\mathbf{A}=\mathbf{B}$$

since the curl of any gradient is zero. Now following (2.25) we write

$$\mathbf{E}'=-\boldsymbol{\nabla}V'-\frac{\partial\mathbf{A}'}{\partial t}=-\boldsymbol{\nabla}V-\boldsymbol{\nabla}\chi-\frac{\partial\mathbf{A}}{\partial t}-\frac{\partial\chi}{\partial t}=\mathbf{E}-\left(\boldsymbol{\nabla}\chi+\frac{\partial\chi}{\partial t}\right)$$

Therefore, the expression in parenthesis must be zero for the electric field to remain unchanged. In terms of special relativity, this means that the "four gradient" of χ must be zero.

2.20 The Poynting Vector $\mathbf{S}=(1/\mu_0)\mathbf{E}\times\mathbf{B}$ for an electromagnetic wave. What is the direction of this vector? Show that the Poynting Vector has the dimensions of "energy flux," that is, energy per unit area per unit time.

Solution
From (2.39), we see that

$$\mathbf{E}_0\times\mathbf{B}_0=\frac{1}{c}\mathbf{E}_0\times(\mathbf{k}_0\times\mathbf{E}_0)=\frac{1}{c}\mathbf{k}_0(\mathbf{E}\cdot\mathbf{E})-\frac{1}{c}\mathbf{E}_0(\mathbf{E}\cdot\mathbf{k}_0)=\frac{1}{c}\mathbf{k}_0E_0^2$$

since $\mathbf{k}_0$ is perpendicular to $\mathbf{E}_0$. Thus, the direction of the Poynting vector is in fact the direction of the wave propagation. (The Poynting vector "points.") Now since

$$\frac{1}{\mu_0}\mathbf{E}_0\times\mathbf{B}_0=\frac{\epsilon_0}{\epsilon_0\mu_0}\mathbf{E}_0\times\mathbf{B}_0=\frac{\epsilon_0E_0^2}{c\epsilon_0\mu_0}\mathbf{k}_0=2c\left(\frac{1}{2}\epsilon_0E_0^2\right)\mathbf{k}_0$$

we see that the Poynting vector has dimensions

$$\text{velocity}\times\text{energy density}=\text{energy}/(\text{area}\times\text{time})$$

(See Problem 2.4. Also, $\mathbf{k}_0$ is a dimensionless unit vector.)

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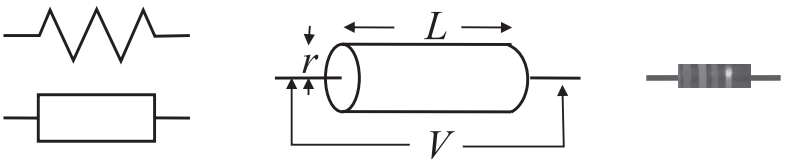


FIGURE 2.1. The resistor as used in electrical circuits. The upper symbol was completely standard,but has more recently been replaced by the lower one. They are used in circuit diagrams regardless of the specific type of resistor used. We can model a resistor as a long cylinder, made of some uniform, isotropic material, with wires attached to the flat faces. The model quite resembles a standard carbon resistor. The colored bands on the resistor are a code for its resistance value and precision standards.

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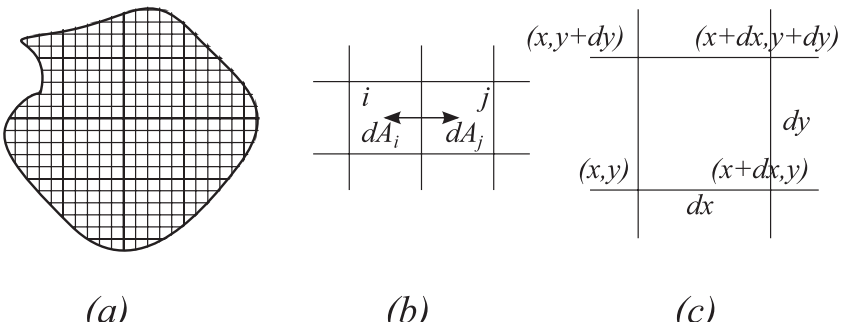


FIGURE 2.2. Geometric explanation of the derivation of the divergence theorem. (a) The volume integral over a region is the sum of small rectangles (or rectangular blocks in three dimensions). (b) Any quantity proportional to the area of a volume element cancels on adjacent cells, leaving only the edges on the boundary. (c) Coordinate system for analyzing the integral over one small cell.

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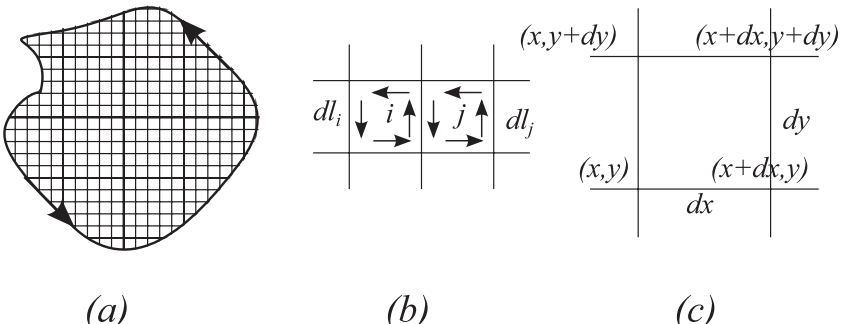


FIGURE 2.3. Geometric explanation of the derivation of Stokes' theorem. (a) The surface integral over a regionis the sum of small rectangles. (b) Any quantity proportional to a loop over the area element cancels on adjacent cells, leaving only the edges on the boundary. (c) Coordinate system for analyzing the integral over one small cell.

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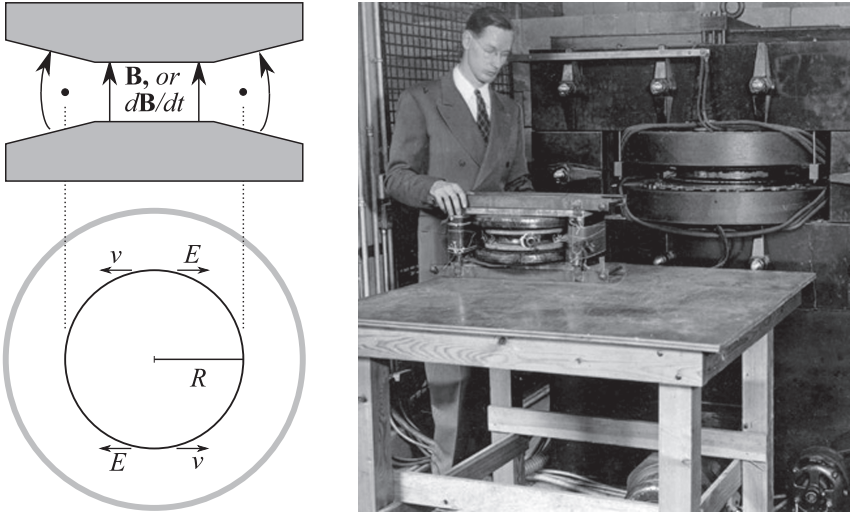


FIGURE 2.4. A Betatron accelerates particles using the electric field from magnetic induction. The diagram shows the principle, and defines quantities that allow us to derive the so-called *Betatron Condition*. The photograph (Courtesy Department of Physics, University of Illinois at Urbana-Champaign) shows early Betatrons with inventor, Donald Kerst.

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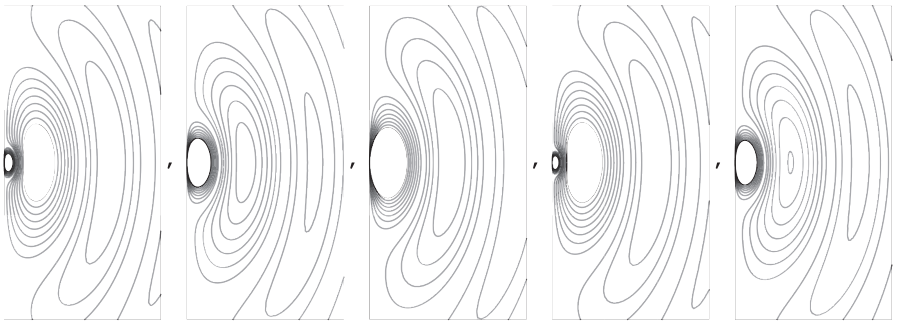


FIGURE 2.5. A schematic representation of electric dipole radiation. The frames from left to right represent increasing time, while each frame represents the same portion of space. The lines are contours of constant electric field strength. The source is an infinitesimally small oscillating electric dipole at the origin, oriented in the vertical direction. The field strength is inversely proportional to distance from the origin, with the field lines forming closed loops that move away from the dipole.

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