Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Manual

SKU: 9781498738606-SOLUTIONS

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.1 Fill in the details leading to Eq. (2.17) and use it to find the weighted residuals formulation of Example 2.1

Solution:

a)

$$\int_0^L \phi(x)\left[-K\frac{d^2T}{dx^2}-Q\right]dx=0 \qquad (Eq.2.13)$$

Integration by parts: $\int_0^L u\,dv = uv\Big|_0^L - \int_0^L v\,du$. Set $u=\phi(x)$ and $dv=-K\frac{d^2T}{dx^2}$, then

$v=-K\frac{dT}{dx}$ and $du=\frac{d\phi}{dx}dx$ and we have

$$\int_0^L \phi(x)\left[-K\frac{d^2T}{dx^2}-Q\right]dx=\phi(x)\left(-K\frac{dT}{dx}\right)_0^L-\int_o^L\left(-K\frac{dT}{dx}\right)\frac{d\phi}{dx}dx$$

$$=\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx-K\phi\frac{dT}{dx}\Big|_0^L$$

Substitute into Eq. 2.13 and we get

$$\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx-\int_o^L \phi Q\,dx-K\phi\frac{dT}{dx}\Big|_0^L=0 \qquad (Eq.2.17)$$

b)

$$T(0)=T_0 \qquad and \qquad -K\frac{dT}{dx}\Big|_{x=L}=h(T-T_\infty)$$

Replacing in Eq. 2.17

$$\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx-\int_o^L \phi Q\,dx-\phi(L)h(T(L)-T_\infty)=0$$

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.2 Use Eq. (2.17) to find the weighted residuals formulation of Eqs. (2.1) – (2.3) in terms of the heat flux q.

Solution:

$$\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx - \int_o^L \phi Q\, dx + \phi\left(-K\frac{dT}{dx}\right)\Big|_0^L = 0$$

The heat flux term at x = L must be zero, and $-K\left.\frac{dT}{dx}\right|_{x=0} = q$ Hence

$$\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx = \int_o^L \phi Q\, dx + \phi(0)\, q$$

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.3 Obtain expression (2.30).

Solution:

The weighted residuals form for the second term is

$$\int_0^L \phi(x)\left[-K\frac{d^2T}{dx^2}-Q\right]dx=0$$

After integration by parts on the first term

$$\int_o^L K\frac{d\phi}{dx}\frac{dT}{dx}dx-\int_o^L \phi Q\,dx+\phi\left(-K\frac{dT}{dx}\right)\Big|_{L/2}^{L}=0$$

Note: The boundary terms are set to zero if no fluxes are prescribed.

The functions $\phi_i(x)$ for element 2 are

$$\phi_2(x)=\frac{2}{L}(L-x)$$

$$\phi_3(x)=\frac{2}{L}(x-L/2)$$

Let $$T(x)=\frac{2}{L}(L-x)a_2+\frac{2}{L}(x-L/2)a_3 \quad (Eq.\ 2.19)$$

Using Eq. 2.28

$$\int_{L/2}^{L}\begin{bmatrix}-\frac{2}{L}\\ \frac{2}{L}\end{bmatrix}\begin{bmatrix}-\frac{2}{L} & \frac{2}{L}\end{bmatrix}dx\begin{Bmatrix}a_2\\ a_3\end{Bmatrix}-Q\int_{L/2}^{L}\begin{Bmatrix}\frac{2}{L}(L-x)\\ \frac{2}{L}(x-L/2)\end{Bmatrix}dx-\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Or

$$\frac{4K}{L^2}\int_{L/2}^{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}dx\begin{Bmatrix}a_2\\ a_3\end{Bmatrix}-\frac{2Q}{L}\int_{L/2}^{L}\begin{Bmatrix}(L-x)\\ (x-L/2)\end{Bmatrix}dx-\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Integrating we have

$$\frac{2K}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}a_2\\ a_3\end{Bmatrix}-\frac{QL}{4}\int_{L/2}^{L}\begin{Bmatrix}1\\1\end{Bmatrix}dx-\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.4 Consider the equation

$$\frac{d^2u}{dx^2}+u+x=0, \qquad 0<x<1$$

with $u(0) = u(1) = 0$. Assume an approximation for $u(x) = a_1\phi_1(x)$ with $\phi_1(x) = x(1 - x)$. The residual is

$$R(u,x)=-2a_1+a_1[x(1-x)]+x$$

and the weight $W_1(x) = \phi_1(x) = x(1 - x)$. Find the solution from the following integral

$$\int_0^1 W(x)R(u,x)dx=0$$

.Solution:

$$\int_0^1 W(x)\,R(u,x)\,dx=\int_0^1 x(1-x)(-2a_1+a_1x(1-x)+x)\,dx=0$$

Or

$$\int_0^1 W(x)\,R(u,x)\,dx=\int_0^1 (-2a_1x+(3a_1+1)x^2-2a_1+1)x^3+a_1x^4)\,dx=0$$

$$-a_1+\frac{1}{3}(3a_1+1)-\frac{1}{4}(2a_1+1)+\frac{1}{5}a_1=-\frac{3}{10}a_1+\frac{1}{12}=0$$

$$a_1=\frac{5}{18} \qquad \text{therefore} \qquad u(x)\approx\frac{5}{18}(x-x^2)$$

Compare with the exact solution given by

$$u^*(x)=\frac{\sin x}{\sin 1}-x$$

Evaluating at x=1/2 $u(1/2)=0.069444$ less than 5% error.

$$u^*(1/2)=0.069747$$

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.5 Use the weak statement formulation to find solutions to the equation

$$\frac{d^2u}{dx^2}=1, \qquad 0<x<1$$

$$u(0)=0, \qquad \frac{du}{dx}\Big|_{x=1}=1$$

(a) Using linear interpolation functions, as explained in Section 2.3, use (i) one element and (ii) two elements.

(b) Using simple polynomial functions that satisfy the boundary conditions at the left-hand side, i.e.,

(i) $u(x) = a_1x$, hence $\phi_1(x) = W_1(x) = x$

(ii) $u(x) = a_1x+a_2x^2$, hence $\phi_1(x) = W_1(x) = x$,

$\phi_2(x) = W_2(x) = x^2$.

(c) Using circular functions that satisfy the boundary conditions at the left-hand side, i.e.,

(i) $u(x) = a_1 \sin(\pi/2Lx)$

(ii) $u(x) = a_1 \sin(\pi/2Lx) + aa_2 \sin(3\pi/2Lx)$

(d) Find the analytical solution and compare with results from (a), (b) and (c).

Solution

a) The residual function is $R(u,x)=\dfrac{d^2u}{dx^2}+1$ The weak form after integration by parts is

$$\int_0^1\left(\frac{dW}{dx}\frac{du}{dx}+W\right)dx-W\frac{du}{dx}\Big|_0^1=0$$

i)

One element solution

$$\phi_1=1-x \qquad \phi_2=x$$

$$u(x)=(1-x)a_1+x\,a_2$$

$$\int_0^1\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}dx\begin{Bmatrix}a_1\\a_2\end{Bmatrix}+\int_0^1\begin{bmatrix}1-x\\x\end{bmatrix}dx-\begin{Bmatrix}0\\1\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Integrating

$$a_1-a_2=-1/2$$
$$-a_1+a_2=1/2$$

since $a_1=0$ then $a_2=1/2$ *or* $u(x)=x/2$

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

ii)

Two element solution. Element 1:

$$\phi_1 = 1-2x \qquad \phi_2 = 2x$$

$$\int_0^{1/2}\begin{bmatrix}-2\\2\end{bmatrix}\begin{bmatrix}-2 & 2\end{bmatrix}dx\begin{Bmatrix}a_1\\a_2\end{Bmatrix}+\int_0^{1/2}\begin{bmatrix}1-2x\\2x\end{bmatrix}dx-\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \quad or \quad \begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}\begin{Bmatrix}a_1\\a_2\end{Bmatrix}=\begin{Bmatrix}-1/4\\-1/4\end{Bmatrix}$$

Element 2

$$\phi_2 = 2(1-x) \qquad \phi_3 = 2(x-1/2)$$

$$\int_{1/2}^{1}\begin{bmatrix}-2\\2\end{bmatrix}\begin{bmatrix}-2 & 2\end{bmatrix}dx\begin{Bmatrix}a_2\\a_3\end{Bmatrix}+\int_{1/2}^{1}\begin{bmatrix}2(1-x)\\2(x-1/2)\end{bmatrix}dx-\begin{Bmatrix}0\\1\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \quad or \quad \begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}\begin{Bmatrix}a_2\\a_3\end{Bmatrix}=\begin{Bmatrix}-1/4\\3/4\end{Bmatrix}$$

Assemble the elements

$$\begin{bmatrix}2 & -2 & 0\\-2 & 4 & -2\\0 & -2 & 2\end{bmatrix}\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix}=\begin{Bmatrix}-1/4\\-1/2\\3/4\end{Bmatrix} \quad \text{since} \quad a_1 = 0 \;\; we\,get \;\; a_2 = 1/8 \;\; and \;\; a_3 = 1/2$$

$$u(x)=\begin{cases} x/4 & 0\le x\le 1/2\\ \dfrac{3x-1}{4} & 1/2\le x\le 1\end{cases}$$

b)

Using polynomials

i)

$$\phi_1 = x \qquad u(x) = a_1 x$$

$\int_0^1 (1\cdot a_1 + x)\,dx - 1 = 0$, $a_1 = 1/2$, $u(x) = x/2$, same as using one element.

ii)

$\phi_1 = x$, $\phi_2 = x^2$, $u(x) = a_1 x + a_2 x^2$

For W=ϕ_1 $\int_0^1 [1\cdot(a_1 + 2a_2 x) + x]dx - 1 = 0$

For W=ϕ_2 $\int_0^1 [2x\cdot(a_1 + 2a_2 x) + x^2]dx - 1 = 0$

The system becomes

$$a_1 + a_2 = 1/2$$

$$a_1 + \frac{4}{3}a_2 = 2/3 \qquad or \qquad a_1 = 0 \;,\; a_2 = 1/2 \;\; and \;\; u(x) = x^2/2$$

c)

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

Using circular functions

i)

$$\phi_1 = \sin\left(\frac{\pi}{2}x\right) \quad , \quad u(x) = a_1 \sin\left(\frac{\pi}{2}x\right) \quad , \quad \phi_1' = \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)$$

$$\int_0^1 \left[\frac{\pi^2}{4}a_1 \cos^2\left(\frac{\pi}{2}x\right) + \sin\left(\frac{\pi}{2}x\right)\right]dx - \sin\left(\frac{\pi}{2}x\right)\frac{du}{dx}\Big|_0^1 = 0$$

Use the identities $\int \cos^2 \alpha x\, dx = \frac{1}{2\alpha}(\sin \alpha x \cos \alpha x - \alpha x)$ and $\int \sin\left(\frac{\pi}{2}x\right)dx = -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)$ .Ther

$$\frac{\pi^2 a_1}{4}\left(\sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}x\right) - \frac{x}{2}\right)\Big|_0^1 + \left(-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\right)\Big|_0^1 - \sin\left(\frac{\pi}{2}x\right)\frac{du}{dx}\Big|_0^1 = 0 \quad or$$

$$\frac{\pi^2 a_1}{8} + \frac{2}{\pi} - 1 = 0 \qquad \text{hence} \quad a_1 \approx 0.295 \quad \text{and} \quad u(x) = 0.295 \sin\left(\frac{\pi}{2}x\right)$$

ii)

$$\phi_2 = \sin\left(\frac{3\pi}{2}x\right) \quad , \quad u(x) = a_1 \sin\left(\frac{\pi}{2}x\right) + a_2 \sin\left(\frac{3\pi}{2}x\right).$$

For $W = \phi_1$:

$$\int_0^1 \left[\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right)\left(a_1 \cos\left(\frac{\pi}{2}x\right) + a_2 \cos\left(\frac{3\pi}{2}x\right)\right) + \sin\left(\frac{\pi}{2}x\right)\right]dx - \sin\left(\frac{\pi}{2}x\right)\frac{du}{dx}\Big|_0^1 = 0$$

For $W = \phi_2$:

$$\int_0^1 \left[3\frac{\pi^2}{4}\cos\left(3\frac{\pi}{2}x\right)\left(a_1 \cos\left(\frac{\pi}{2}x\right) + a_2 \cos\left(\frac{3\pi}{2}x\right)\right) + \sin\left(\frac{3\pi}{2}x\right)\right]dx - \sin\left(\frac{3\pi}{2}x\right)\frac{du}{dx}\Big|_0^1 =$$

$$\frac{\pi^2}{8}a_1 + 0 \cdot a_2 + \frac{2}{\pi} - 1 = 0 \quad (a_1 \text{ is unchanged})$$

$$0 \cdot a_1 + \frac{9\pi^2}{8}a_2 + \frac{2}{3\pi} + 1 = 0 \quad , \quad a_2 \approx -0.109 \quad and$$

$$u(x) \approx 0.295 \sin\left(\frac{\pi}{2}x\right) - 0.109 \sin\left(\frac{3\pi}{2}x\right)$$

d) Analytical solution

$\frac{d^2u}{dx^2} = 1$,integrating $u(x) = \frac{x^2}{2} + c_1 x + c_2$ and applying boundary conditions $u^*(x) = \frac{x^2}{2}$.

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

Notice that all the approximations are exact at x = 1/2, and the Galerkin approximations using x and x^2 both yield the exact solution. Also notice that the boundary condition at x = 1 is satisfied only approximately. In case a), the first solution gives $\left.\frac{du}{dx}\right|_{x=1} = \frac{1}{2}$ and the second solution gives $\left.\frac{du}{dx}\right|_{x=1} = \frac{3}{4}$ which is getting better.This point is discussed in more detail in problem 2.6.

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.6 Subdivide the interval $0 \le x \le L$ into 10 linear elements, construct the element equations for four consecutive elements starting with element four, and show that none of the parameters a_i is related to more than two additional parameters in the equations. The only ones involved being a_{i-1} and a_{i+1} (as a consequence of this, the final system of linear equations will involve a tri-diagonal matrix which is easy to solve).

Solution:

For an element $e_k = \{x / x_k \le x \le x_{k+1}\}$ the element stiffness matrix is obtained from $\int_{x_k}^{x_{k+1}} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx$. All the stiffness matrices must be equal, so let's calculate the first one.

$$\phi_1 = 1 - \frac{10}{L}x \qquad \phi_2 = \frac{10}{L}x \quad thus \quad \int_0^{L/10} K \begin{bmatrix} -\frac{10}{L} \\ \frac{10}{L} \end{bmatrix} \begin{bmatrix} -\frac{10}{L} & \frac{10}{L} \end{bmatrix} dx = \frac{10K}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and after assembling $$\frac{10K}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_9 \\ a_{10} \\ a_{11} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note that each equation i involves only three unknowns a_{i-1}, a_i *and* a_{i+1}, so the stiffness matrix is tri-diagonal.

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.7 The Rayleigh-Ritz formulation for the problem of solving Eq. (2.1) with boundary conditions (2.2) and (2.3) can be stated as: Minimize the functional

$$F(T)=\int_0^L\left\{\frac{K}{2}\left[\frac{dT}{dx}\right]^2-QT\right\}dx-Tq\Big|_{x=0}$$

over all functions $T(x)$ with square integrable first derivatives that satisfy Eq. (2.3) at $x = L$. Approximate $T(x)$ using two linear elements as we did before and replace into Eq. (2.1) to obtain $F(T) \cong F(a_1, a_2, a_3)$. Now minimize $F(a_1, a_2, a_3)$ as a function of three variables. Show that the final system of equations is identical to Eq. (2.31), thus, in this case, the Galerkin method and the Rayleigh-Ritz method are equivalent.

Solution:

Element1:

$$\phi_1=1-\frac{2}{L}x\qquad \phi_2=\frac{2}{L}x$$

$$\int_0^{L/2}\frac{K}{2}\left(-\frac{2}{L}a_1+\frac{2}{L}a_2\right)^2-Q\left[\left(1-\frac{2}{L}x\right)a_1+\frac{2}{L}xa_2\right]dx-a_1q=\frac{K}{L}(a_2-a_1)^2-\frac{QL}{4}(a_1+a_2)-a_1q$$

(1)

Element2:

$$\phi_2=2-\frac{2}{L}x\qquad \phi_3=\frac{2}{L}x-1$$

$$\int_{L/2}^{L}\frac{K}{2}\left(-\frac{2}{L}a_2+\frac{2}{L}a_3\right)^2-Q\left[\left(2-\frac{2}{L}x\right)a_2+\left(\frac{2}{L}x-1\right)a_3\right]dx-a_1q=\frac{K}{L}(a_3-a_2)^2-\frac{QL}{4}(a_2+a_3)$$

(2)

$$F(T)=(1)+(2)=\frac{K}{L}(a_2-a_1)^2+\frac{K}{L}(a_3-a_2)^2-\frac{QL}{4}(a_1+2a_2+a_3)-a_1q$$

$$F(T)=F(a_1,a_2,a_3)=\frac{K}{L}\left(a_1^2-2a_1a_2+2a_2^2-2a_2a_3+a_3^2\right)$$

The conditions for a minimum are $\dfrac{\partial F}{\partial a_1}=\dfrac{\partial F}{\partial a_2}=\dfrac{\partial F}{\partial a_3}=0$ thus

$$\frac{\partial F}{\partial a_1}=\frac{2K}{L}(a_1-a_2)-\frac{QL}{4}-q=0$$

$$\frac{\partial F}{\partial a_2}=\frac{2K}{L}(-a_1+2a_2-a_2)-\frac{QL}{2}=0$$

$$\frac{\partial F}{\partial a_3}=\frac{2K}{L}(a_3-a_2)-\frac{QL}{4}=0$$

In Matrix form $\dfrac{2K}{L}\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix}=\dfrac{QL}{4}\begin{Bmatrix}1\\2\\1\end{Bmatrix}+\begin{Bmatrix}q\\0\\0\end{Bmatrix}$ which is identical to Eq. 2.31.

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.8 Let us consider once more Problem 2.3 above. Assume that $u(x) = a_1x^2$ and $W_1(x) = x^2$, but this time use Eq. (2.11) and proceed as in Problem 2.2 in order to obtain a solution. Show that this solution is identical to that obtained in Problem 2.3(b)(ii), and hence, the process of integration by parts does not introduce any changes.

Solution:

$$\int_0^1 \left[W_1 \left(\frac{d^2u}{dx^2} - 1 \right) \right] dx = \int_0^1 x^2 (2a - 1) dx = 0 \quad , \quad \left(\frac{2}{3} x^3 a_1 - \frac{x^3}{3} \right) \Big|_0^1 = 0 \quad , \quad \frac{2}{3} a - \frac{1}{3} = 0 \quad , \quad a_1 = 1/2 \quad , \quad u(x) = \frac{x^2}{2}$$

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.9 In example 2.2, first find the exact analytical solution, then calculate two more Galerkin approximations using

Solution:

The analytical solution is $u(x)=\frac{1}{2}\left(x-x^2\right)$. The Galerkin approximation is

$$\int_0^1\left(\frac{dW}{dx}\frac{du}{dx}-W\right)dx=0.$$

$$u(x)=a_1\sin\pi x+a_2\sin 2\pi x\ ,\ W_1=\sin\pi x\ ,\ W_2=\sin 2\pi x$$

a)

$$W_1:\ \int_0^1\left[\pi^2\cos\pi x\left(a_1\cos\pi x+2a_2\cos 2\pi x\right)-\sin\pi x\right]dx=0$$

$$W_2:\ \int_0^1\left[2\pi^2\cos 2\pi x\left(a_1\cos\pi x+2a_2\cos 2\pi x\right)-\sin 2\pi x\right]dx=0 \quad or$$

$$\frac{\pi^2}{2}a_1+0\cdot a_2-\frac{2}{\pi}=0\ ,\ a_1=\frac{4}{\pi^3}\quad ,\ and\quad 0\cdot a_1+2\pi^2a_2+0=0\ ,\ a_2=0$$

$$u(x)=\frac{4}{\pi^3}\sin\pi x$$

b) $u(x)=a_1\sin\pi x+a_2\sin 2\pi x+a_3\sin 3\pi x\ ,\ W_1=\sin\pi x\ ,\ W_2=\sin 2\pi x\ ,\ W_3=\sin 3\pi x$

We only need to calculate a_3, a_1 *and* a_2 remain the same due to the orthogonality of the sine functions.

$$W_3:\ \int_0^1\left[3\pi^2\cos 3\pi x\left(a_1\cos\pi x+2a_2\cos 2\pi x+3a_3\cos 3\pi x\right)-\sin 3\pi x\right]dx=0$$

$$\frac{9\pi^2}{2}a_3-\frac{2}{3\pi}=0\ ,\ a_3=\frac{4}{27\pi^3}\ ,\ u_3(x)=\frac{4}{3\pi^3}\left(\sin\pi x+\frac{1}{27}\sin 3\pi x\right)$$

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.10 Derive Eq. (2.32) from Eq. (2.31).

Eqs. 2.32 $a_1 - a_2 = \frac{qL}{2K} + \frac{QL^2}{8K}$, $-a_1 + 2a_2 = \frac{QL^2}{4K} + T_L$ adding $a_2 = \frac{qL}{2K} + \frac{3QL^2}{8K} + T_L$ and

substituting into the first equation $a_1 = \frac{qL}{K} + \frac{QL^2}{2K} + T_L$

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.11 Fill in the details in Example 2.3 and verify the solution.

Solution:

The weak form is $\int_0^1 \left(\frac{d\phi}{dx}\frac{dT}{dx} + W \right) dx - \int_0^1 \phi\, dx = 0$ Use the Galerkin approximation $u(x) = a_1 \sin \pi$, $\phi(x) = \sin \pi x$, $du = a_1 \pi \cos \pi x\, dx$, $d\phi = \pi \cos \pi x\, dx$. Replace in the integrals to get

$\int_0^1 (\pi \cos \pi x)(a_1 \pi \cos \pi x)\, dx = \int_0^1 \sin \pi x\ dx$ and using the trigonometric identity

$\int_0^1 \cos^2 \pi x\, dx = \int_0^1 \frac{1}{2}(1 + \cos 2\pi x)\ dx$ we get $a_1 = \frac{4}{\pi^3}$.

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.12 Show that if the expressions for ϕ_i given by Eq. (2.20) are used in Eq. (2.19), then $a_i = T_i$.

Solution:

Eq. 2.19 is $T(x) = \phi_i(x)a_i + \phi_{i+1}(x)a_{i+1} \quad x_i \le x \le x_{i+1}$ where

$$\phi_i(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} \quad , \; and \quad \phi_{i+1}(x) = \frac{x - x_i}{x_{i+1} - x_i}$$

Substituting $T(x) = \left[\frac{x_{i+1} - x}{x_{i+1} - x_i}\right] a_i + \left[\frac{x - x_i}{x_{i+1} - x_i}\right] a_{i+1}$ and evaluating at the nodes

$T(x_i) = a_i = T_i \;\; and \;\; T(x_{i+1}) = a_{i+1} = T_{i+1}$.

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.13 Find the weak formulation of Eq. (2.1) with the boundary condition

$$-K\left.\frac{dT}{dx}\right|_{x=0} = h\left(T - T_\infty\right) \text{ and } -K\left.\frac{dT}{dx}\right|_{x=1} = q\,.$$

Solution:

The general form is given in Eq. 2.17 $\int_0^L \frac{d\phi}{dx}\frac{dT}{dx}dx - \int_0^L \phi Q\,dx - K\,\phi\left.\frac{dT}{dx}\right|_o^L = 0$

Substituting the flux boundary conditions we get

$$\int_0^L \frac{d\phi}{dx}\frac{dT}{dx}dx - \int_0^L \phi Q\,dx + \phi(L)q - \phi(0)\,h(T(0) - T_\infty) = 0$$

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma

2.14 Using two linear elements solve the equations in problem (2.13) with $L = 1\ m$, $K = 200\ w/mK$, $Q = 100\ W/m^3$, $h = 150\ W/m^2\ K$, $T_\infty = 100C$ and $q = 2{,}000\ W/m^2$

Solution:

Using Eqs. 2.19 and 2.30 $\dfrac{2K}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix}-\dfrac{QL}{4}\begin{Bmatrix}1\\ 1\end{Bmatrix}-h\begin{Bmatrix}T_1-T_\infty\\ 0\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$ Substitute the parameters values 1) $\begin{bmatrix}250 & -400\\ -400 & 400\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix}=\begin{Bmatrix}-14750\\ 25\end{Bmatrix}$ 2) $\begin{bmatrix}400 & -400\\ -400 & 400\end{bmatrix}\begin{Bmatrix}T_2\\ T_3\end{Bmatrix}=\begin{Bmatrix}25\\ 25-q\end{Bmatrix}$

assembling

1) $\begin{bmatrix}250 & -400 & 0\\ -400 & 800 & -400\\ 0 & -400 & 400\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix}=\begin{Bmatrix}-14750\\ 50\\ 25-q\end{Bmatrix}$ With solution

$$T_1 = 97.833 + 0.0066667q$$
$$T_2 = 98.021 + 0.0041667q$$
$$T_3 = 98.084 + 0.0016667q$$

Finite Element Method Basic Concepts and Applications with MATLAB MAPLE and COMSOL 3rd edition Pepper Solutions Ma