# *Differential Equations for Engineers The Essentials 1st edition Kalbaugh Solutions Manual*

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| Date | Class # | # of slides | Topics | Reading Assigned | Problems Assigned | Problems Reviewed |
|---|---|---|---|---|---|---|
| | 1 | 36 | Administrative matters, course objectives, importance of DEs to engineers, review of math foundations, classes of DEs | Chap 1 (14 pages) | | _ |
| | 2 | 31 | First order linear ODEs: RC circuit; general solution to homogeneous eqns; in-class homogeneous problems; water pipe temperature example; general solution nonhomogeneous; In-class nonhomogeneous problems | Chap 2 (11 pages) | | _ |
| | 3 | 35 | 1st order linear ODEs: System viewpt First order nonlinear separable ODEs: General solution approach; In-class problems; Sounding rocket phases 1 & 2 | Chap 3 thru Sec 3.1 (7 pages) | | From Class #1 |
| | 4 | 21 | First order nonlinear separable ODEs: Sounding rocket phase 3; Short quiz #1 | Chap 3, Sec 3.2, 3.3 (5 pages) | | From Class #2 |
| | 5 | 32 | Review of short quiz #1 First order ODEs: successive approximations with example; in-class problems; existence and uniqueness | Chap 4, thru Sec 4.4 (9 pages) | | From Class #3 |
| | 6A/B | 30/36 | Qualitative analysis Stability revisited Computing project phase 1 | Chap 4, Sec 4.5, 4.6 (3 pages) | Computing project phase 1 | From Class #4 |
| | 7 | 24 | 2nd order LTI homogeneous ODEs LRC circuit, characteristic equation, real and repeated roots; in-class problems | Chap 5 thru Sec 5.1.2 (6 pages) | | From Class #5 |
| | 8 | | Test #1 | | | _ |
| | 9 | 28 | Review of Test #1 2nd order LTI homogeneous ODEs, LRC circuit, complex roots, fundamental solutions, In-class example problems | Chap 5, Sec 5.1.3, 5.1.4, 5.2 (5 pages) | | From Class #7 |
| | 10 | 41 | 2nd order LTI Nonhomogeneous ODEs; Cruise control Undetermined coefficients method LRC circuit with sine source | Chap 5, Sec 5.3, 5.4 (15 pages) | | From Class #6 |
| | 11 | 42 | Process & example for kernel method Undetermined coefficients example Higher order ODEs | Chap 6 (10 pages) | | From Class #9 |

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|---|---|---|---|---|---|---|
| | | | Satellite orbit decay | | | |
| | 12 | | Review for Midterm | | | From Classes #10, 11 |
| | 13 | | Mid-term exam | | | _ |
| | 14A/B | 42/44 | Review of mid-term<br>Laplace transforms: intro, homogeneous equations<br>Computing project phase 2 | Chap 7 thru Sec 7.5 (11 pages) | Computing project phase 2 | |
| | 15 | 46 | Laplace transforms: Nonhomogeneous equations et al | Chap 7, Sec 7.6 (20 pages) | | |
| | 16 | 43 | State space format: Numerical methods<br>Review of matrix algebra<br>Linear systems in state space format | Chap 8 thru Sec 8.4 (11 pages) | | |
| | 17 | 20 | State space format: Heat transfer<br>Short quiz #2 | Chap 8, Sec 8.5.1.1 (8 pages) | | From class #15 |
| | 18 | 35 | Review of short quiz #2<br>State space format: Two-state electrical circuit; Aircraft dynamics | Chap 8, Sec 8.5.1.2, 8.5.1.3 (15 pages) | | From classes #14,#16 |
| | 19 | 42 | Three state electrical circuit<br>Repeated eigenvalues; Coordinate systems: Vehicle suspension system | Chap 8, Sec 8.5.1.4 thru 8.5.1.7 (12 pages) | | From classes #17,#18 |
| | 20 | | Test #2 | | | _ |
| | 21 | 37 | Review of Test #2<br>Coordinate systems; state transition matrix; nonhomogeneous equations, kernel method, 2-state electrical circuit example | Chap 8, Sec 8.5.1.5 thru 8.5.2.1 (15 pages) | | From class #19 |
| | 22 | 35 | Nonhomogeneous equations: Laplace transform method, trial and error method, PDEs: IV heat equation | Chap 8, Sec 8.5.2.2, 8.5.2.3 (6 pages) | | |
| | 23 | 24 | PDEs: BV heat equation – Fourier series<br>Short quiz #3 | Chap 9 thru Sec 9.1 (11 pages) | | From class #21 |
| | 24 | 35 | Review of Short quiz #3<br>PDEs: wave equation; IV wave equation, BV problem: membrane – power series I | Chap 9, Sec 9.2 thru 9.2.3 (16 pages) | | From class #22 |
| | 25 | 35 | PDEs: higher order Bessell functions; | Chap 9, | | From class |

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|---|---|---|---|---|---|---|
| | | | power series II | Sec 9.2.4 thru 9.2.6 (8 pages) | | #23 |
| | 26 | 34 | PDEs: BV potential equation – Legendre's eq'n; cantilever beam | Chap 9, Sec 9.3, 9.4 (13 pages) | | From class #24, #25 |
| | 27 | | Test #3 | | | |
| | 28 | | Review of Test #3<br>Review for final exam | | | |
| | 29 | | Review for final exam | | | |
| | | | Final Exam | | | |

Differential Equations for Engineers The Essentials 1st edition Kalbaugh Solutions Manual

# *Differential Equations for Engineers: the Essentials*

## *Supplement to Class 4 Notes*

Differential Equations for Engineers The Essentials 1st edition Kalbaugh Solutions Manual

# *Contents of Supplement*

**Differences Between Linear and Nonlinear ODEs in Their Input / Output Response**

**Example: Nonlinear Circuit**

# *Differences Between Linear and Nonlinear ODEs in Their Input / Output Response*

# *Input / Output Characteristics for a Linear System*

**In a linear system of any order (time-varying or time-invariant):**

**If the output is $y_1(t)$ when the input is $u_1(t)$ and the output is $y_2(t)$ when the input is $u_2(t)$**
**then the output is $ay_1(t)+by_2(t)$ when the input is $au_1(t)+bu_2(t)$ for any constants $a$ and $b$**

**In a linear time-invariant system of any order, after transients have died away, when the input is a sine-wave of a given frequency the output is a steady-state oscillation of only that frequency.**

**These characteristics are not generally true of nonlinear systems**

# *Existence and Uniqueness of Solutions to Linear ODEs*

**Theorem: Given a linear nth order ODE:**

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^n y}{dt^n} + \ldots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = g(t)$$

**with initial conditions**

$$\frac{d^{n-1} y}{dt^{n-1}}(0) = y_0^{(n-1)} \qquad \frac{d^{n-2} y}{dt^{n-2}}(0) = y_0^{(n-2)} \qquad \ldots \qquad \frac{dy}{dt}(0) = y_0' \qquad y(0) = y_0$$

**If the coefficients $a_i(t)$ and the input $g(t)$ are continuous for all $t$ then there exists a unique solution to the ODE satisfying the initial conditions for all $t$ .**

**(Stated without proof.)**

**This is not generally true of nonlinear ODEs.**

# *Key Points from the Following Nonlinear Example*

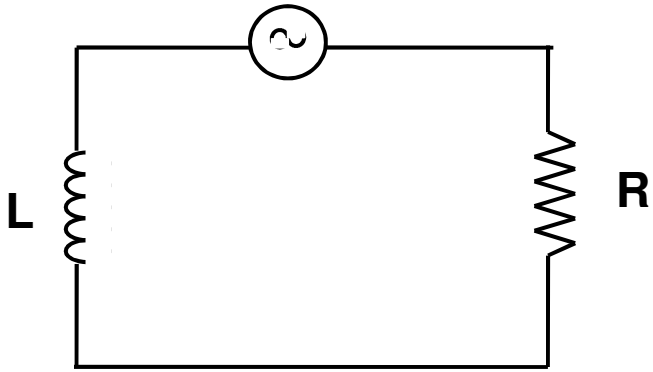
**The method of successive approximations (solving a sequence of linear equations to approximate the solution of a nonlinear equation) is a powerful tool.**

**Nonlinearities in systems designed to be linear cause distortions in the frequency response, introducing "harmonics" (oscillations that are multiples of the input frequency).**

# *Nonlinear Example: LR Circuit*

**Kirchhoff's Law:**

**Sum of voltage drops around a closed circuit = 0**




**Voltage drop over an inductor:**

$$V = L\frac{dI}{dt}$$

**Voltage drop over a resistor:**

$$V = IR$$

**Voltage drop over source:**

$$V = -V_0 \sin(\omega t)$$

**Resulting equation :**

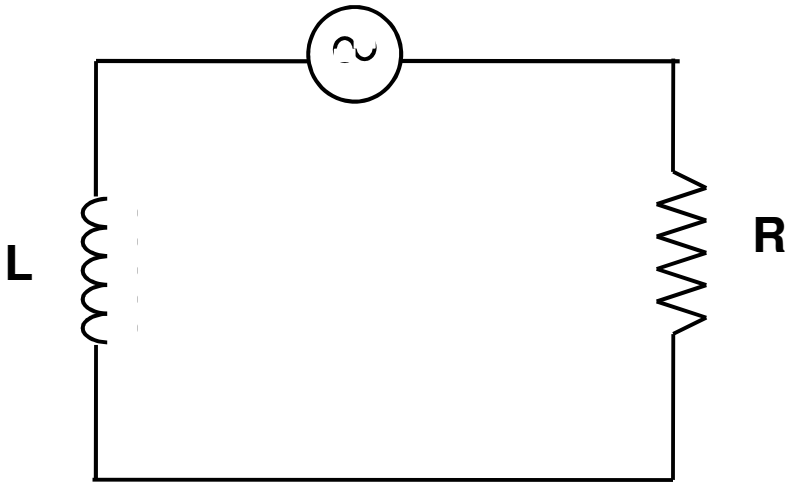
$$L\frac{dI}{dt} + IR = V_0 \sin(\omega t)$$

# Nonlinear Example: LR Circuit (2)

**The solution to**

$$L\frac{dI}{dt} + IR = V_0 \sin(\omega t)$$

$$I(0) = 0$$




**is**

$$I(t) = \frac{\lambda}{\sqrt{\lambda^2 + \omega^2}}(V_0 / R)\sin(\omega t - \theta_1) + \frac{\lambda\omega}{\lambda^2 + \omega^2}(V_0 / R)e^{-\lambda t}$$

**where**

$$\theta_1 = \arctan(\omega / \lambda) \qquad \lambda = R / L$$

# *Nonlinear Example: Recalling the LR Circuit (3)*

**Input / output response:**

**The "input" to the "system" is the voltage source. The "output" is the current (or voltage) over the resistor.**

**Ignoring the transient, the system passes the sine wave frequency perfectly - it introduces no other frequencies.**

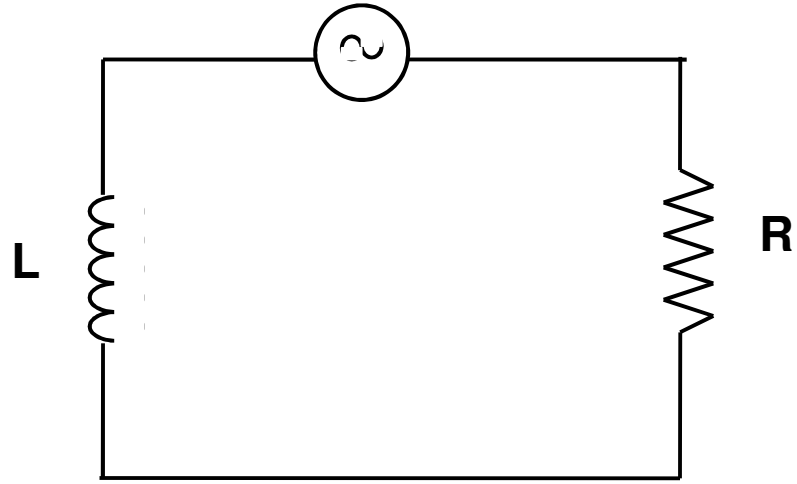
**We only discuss frequency response in the context of time-invariant systems.**

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# What if the Resistor is Slightly Nonlinear?

**Instead of**

$$L\frac{dI}{dt} + IR = V_0 \sin(\omega t)$$



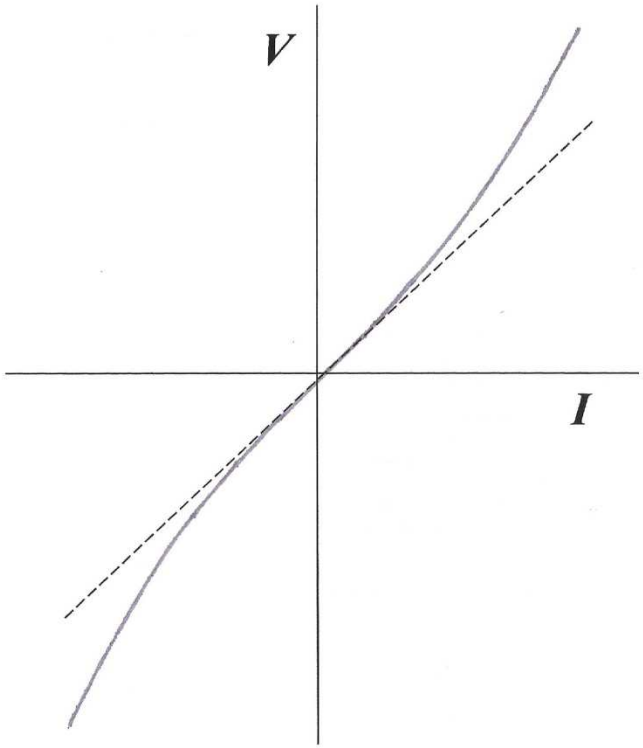

**suppose we have**

$$L\frac{dI}{dt} + R(I + \varepsilon I^3) = V_0 \sin \omega t$$

$$\frac{dI}{dt} + \lambda(I + \varepsilon I^3) = \lambda\left(\frac{V_0}{R}\right)\sin(\omega t) = \lambda I_0 \sin(\omega t)$$

**Equation 1**

# *What if the Resistor is Slightly Nonlinear? (2)*




**Nonlinear resistor characteristic**

# Approach to Solution of Slightly Nonlinear ODE

**Since $\varepsilon$ is small, consider the successive approximations**

$$\frac{dI_1}{dt} + \lambda I_1 = \lambda I_0 \sin(\omega t)$$
$$I_1(0) = 0$$

**Equation 2**

$$\frac{dI_2}{dt} + \lambda I_2 = \lambda I_0 \sin(\omega t) - \varepsilon \lambda I_1^3(t)$$
$$I_2(0) = 0$$

**Equation 3**

$$\frac{dI_3}{dt} + \lambda I_3 = \lambda I_0 \sin(\omega t) - \varepsilon \lambda I_2^3(t)$$
$$I_3(0) = 0$$

**Equation 4**

# *Approach to Solution of Slightly Nonlinear ODE (2)*

**We are solving the nonlinear Equation 1 approximately by the sequential solution of a series of linear Equations 2, 3 and 4.**

**The general solution of**

$$\frac{dI}{dt} + \lambda I = g(t) \qquad \text{is} \qquad I(t) = \int_0^t e^{-\lambda(t-\tau)} g(\tau) d\tau \qquad \text{Equation 5}$$

$$I(0) = 0$$

**Using Equation 5, we have already found that the solution to Equation (2) is**

$$I_1(t) = \frac{\lambda}{\sqrt{\lambda^2 + \omega^2}} I_0 \sin(\omega t - \theta_1) + \frac{\lambda\omega}{\lambda^2 + \omega^2} I_0 e^{-\lambda t}$$

**In what follows we will ignore the transient. We are examining the steady state frequency response.**

# Approach to Solution of Slightly Nonlinear ODE (3)

**Repeating Equation 3:**

$$\frac{dI_2}{dt} + \lambda I_2 = \lambda I_0 \sin \omega t - \lambda \varepsilon I_1^3$$

**This has solution**

$$I_2(t) = \int_0^t e^{-\lambda(t-\tau)} \left( \lambda I_0 \sin(\omega \tau) - \lambda \varepsilon I_1^3 \right) d\tau$$

**Now**

$$\int_0^t e^{-\lambda(t-\tau)} \lambda I_0 \sin(\omega \tau) d\tau = I_1(t)$$

**so**

$$I_2(t) = I_1(t) - \lambda \varepsilon \int_0^t e^{-\lambda(t-\tau)} I_1^3(\tau) d\tau$$

# Approach to Solution of Slightly Nonlinear ODE (4)

**Continuing in this way, we find**

$$I_{n+1}(t) = I_1(t) - \lambda\varepsilon\int_0^t e^{-\lambda(t-\tau)} I_n^3(\tau)\,d\tau$$

**which one can show can be written as**

$$I_n(t) = I_1(t) - G_1 I_0 \eta_n(t)$$

**where**

$$G_1 = \frac{\lambda}{\sqrt{\lambda^2+\omega^2}} \qquad I_0 = V_0 / R$$

$$\eta_n(t) = \sum_{m=1}^{N_n} k_{mn} \sin(m\omega t - \phi_{mn})$$

**Note the higher frequencies – the harmonics**

# *Approach to Solution of Slightly Nonlinear ODE (4)*

***Challenge problem*:**

**Given the steady state solution**

$$I_1(t) = G_1 I_0 \sin(\omega t - \theta_1)$$

**find the steady state component of** $I_2(t)$

# *Differential Equations for Engineers: the Essentials*

## *Class 2 notes*

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# *Agenda: Class 2*

**First order linear differential equations:**

**(1) Engineering example: RC circuit**
**(2) General solution of homogeneous equation**
**(3) In-class homogeneous problems**
**(4) Example: Exposed water pipe in cyclical air temperature**
**(5) General solution of nonhomogeneous equation**
**(6) In-class nonhomogeneous problems**

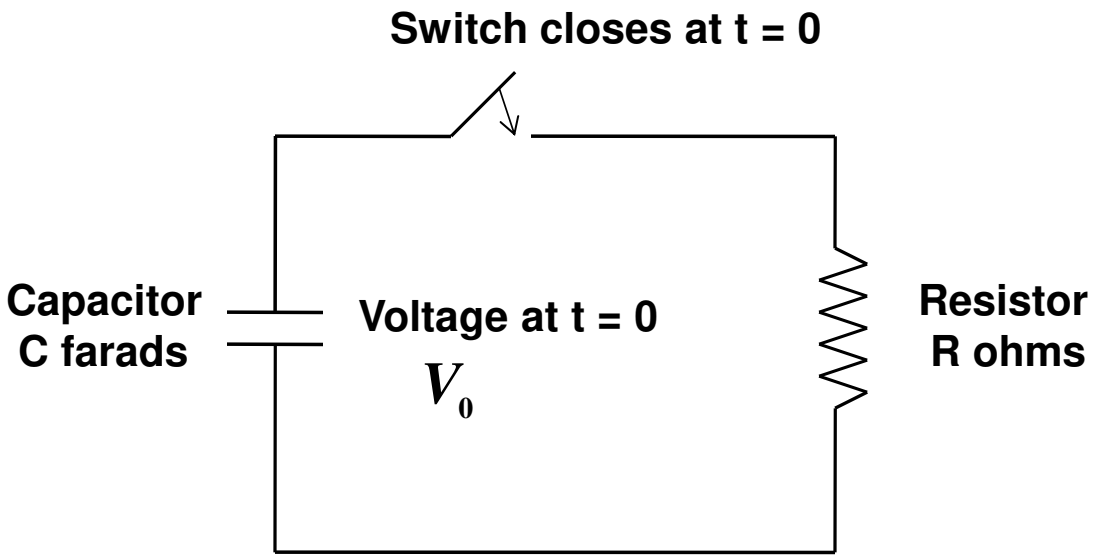
**Homework Assignment 2**

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# *First Order Linear Differential Equations*

## *Example: The RC Electrical Circuit*

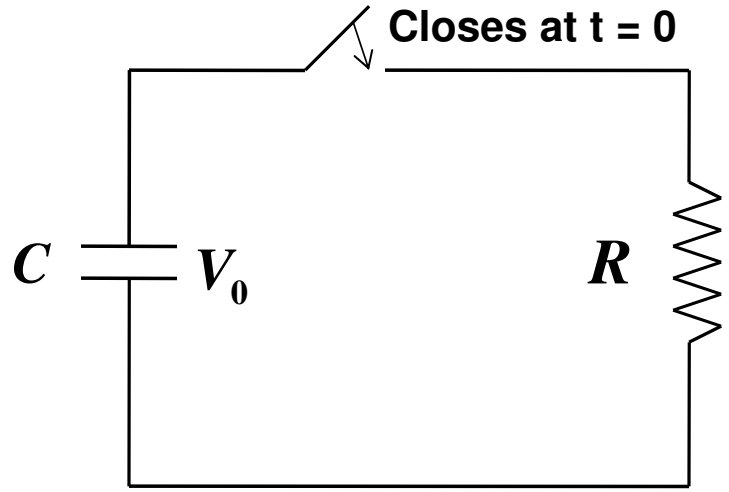
# *Example: RC Electrical Circuit*



# Example: RC Circuit (2)

**Kirchhoff's Law:**

**Sum of voltage drops around a closed circuit = 0**




**Voltage drop over a capacitor:**

$V = q/C$ $q$= **charge on capacitor**

**Voltage drop over a resistor:**

$V = IR$ $I =$ **current through resistor**

**Conservation of electrical charge:**

$$\frac{dq}{dt} = I$$

# Example: RC Circuit (3)

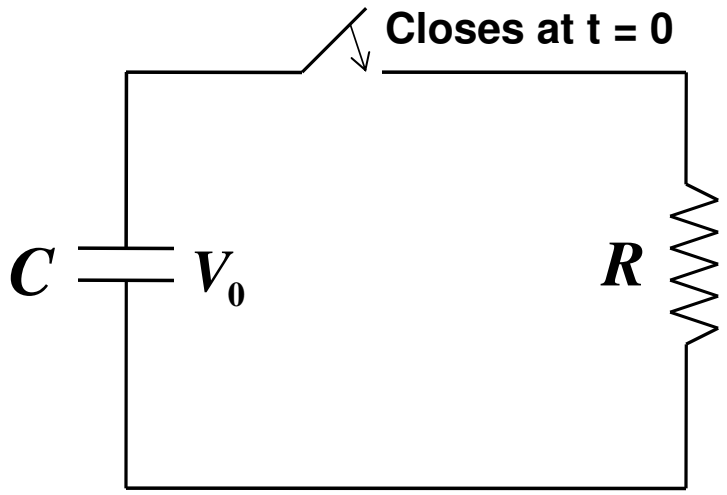
**Resulting differential equation**

$$IR + V = 0$$

$$\frac{dq}{dt}R + V = 0$$

$$RC\frac{dV}{dt} + V = 0$$

$$\frac{dV}{dt} + \frac{1}{RC}V = 0$$




**Initial condition:**

$$V(0) = V_0$$

# *Example: RC Circuit (4)*

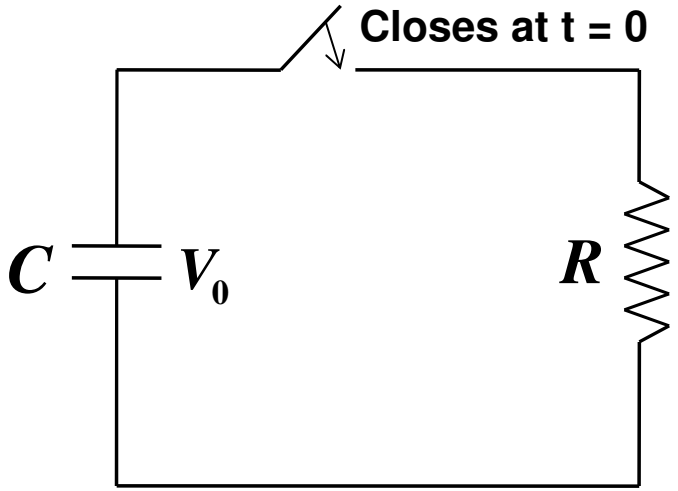
**This is a linear first order ODE. To solve it, we separate variables: i.e., we put all terms involving *V* on the left side and all terms involving *t* on the right: specifically, we divide by *V,* move *1/RC* to the right side and multiply by *dt* :**

$$\frac{dV}{dt} + \frac{1}{RC}V = 0$$

$$\frac{1}{V}\frac{dV}{dt} + \frac{1}{RC} = 0$$

$$\frac{1}{V}\frac{dV}{dt} = -\frac{1}{RC}$$

$$\frac{dV}{V} = -\frac{1}{RC}dt$$



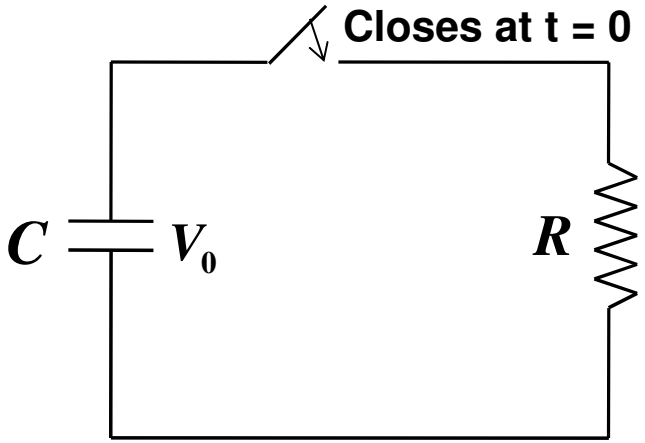

**The text justifies this short-cut procedure**

# Example: RC Circuit (5)

**Next, we integrate both sides:**

$$\int_{V(0)}^{V(t)} \frac{dV}{V} = -\int_0^t \frac{dt}{RC}$$

$$\ln\left(\frac{V(t)}{V(0)}\right) = -t/RC$$




**Taking the exponential of both sides:**

$$\exp\left(\ln\left(\frac{V(t)}{V(0)}\right)\right) = \exp(-t/RC)$$

$$\frac{V(t)}{V(0)} = e^{-t/RC}$$

# Example: RC Circuit (6)

**Hence:**

$$V(t) = V(0)e^{-t/RC}$$

**We require that the voltage over the capacitor at time 0 be given by**

$$V(0) = V_0$$

**and so**

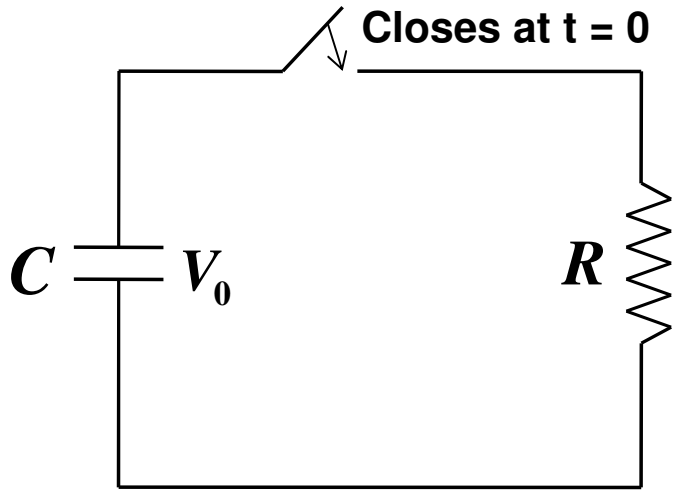
$$V(t) = V_0 e^{-t/RC}$$

# Example: RC Circuit (7)

**In summary, the solution to**

$$\frac{dV}{dt} + \frac{1}{RC}V = 0$$

$$V(0) = V_0$$

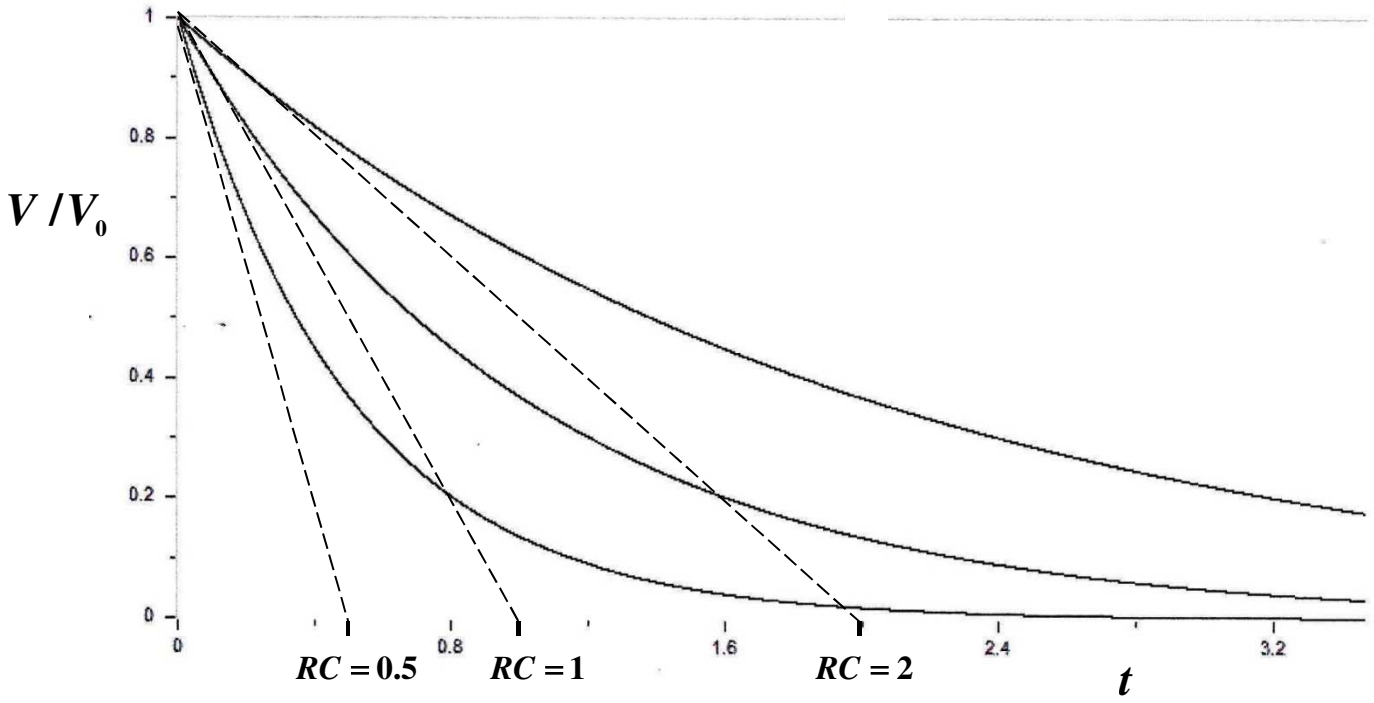



**is**

$$V(t) = V_0 e^{-t/RC}$$

**The product $RC$ has the dimension of time and is called the time constant for the circuit**

# Example: RC Circuit (8)



# *First Order Linear Differential Equations*

## *General Solution of Homogeneous Equation*

# General Solution of Homogeneous Equation

**The general form of a first order linear time-varying ordinary differential equation (ODE) is**

$$\frac{dy}{dt} + p(t)\,y = 0 \qquad y(t_0) = y_0 \qquad \textbf{Equation 1}$$

**How do we solve it?**

# General Solution of Homogeneous Equation (2)

**We separate variables**

$$\frac{dy}{y} = -p(t)dt$$

**Integrate both sides**

$$\int_{y_0}^{y(t)} \frac{dy}{y} = -\int_{t_0}^{t} p(\tau)d\tau$$

$$\ln\left(\frac{y(t)}{y_0}\right) = -\int_{t_0}^{t} p(\tau)d\tau$$

**Take the exponential of both sides**

$$\exp\left(\ln\left(\frac{y(t)}{y_0}\right)\right) = \exp\left(-\int_{t_0}^{t} p(\tau)d\tau\right)$$

$$\frac{y(t)}{y_0} = \exp\left(-\int_{t_0}^{t} p(\tau)d\tau\right)$$

# *General Solution of Homogeneous Equation (3)*

**Summary:**

**The solution of**

$$\frac{dy}{dt} + p(t)\,y = 0 \qquad\qquad y(t_0) = y_0$$

**is**

$$y(t) = y_0 \exp\left(-\int_{t_0}^{t} p(\tau)\,d\tau\right)$$

**Success depends entirely on being able to do the integral**

# Homogeneous First Order Linear ODEs: In-class problems

$$\frac{dy}{dt} + ky = 0$$
$$y(0) = a$$

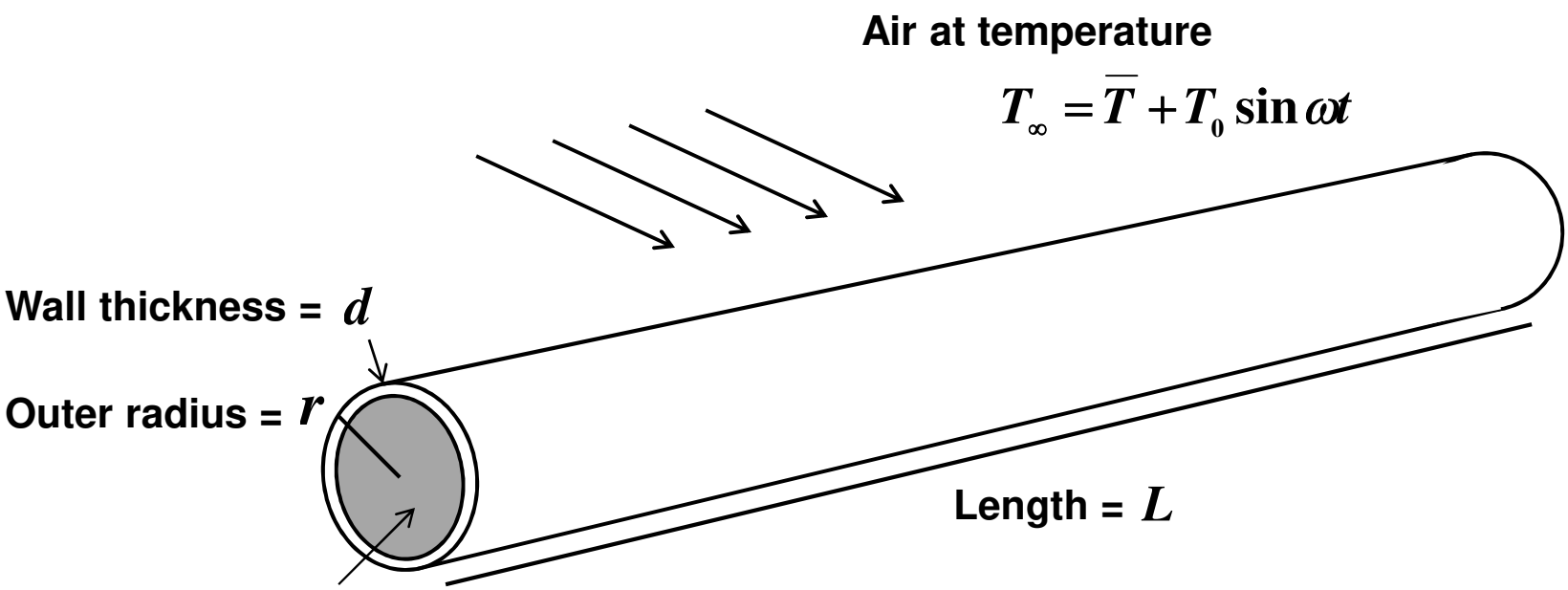
$$\frac{dy}{dt} + ty = 0$$
$$y(0) = b$$

$$\frac{dy}{dt} + \left(\frac{1}{t}\right)y = 0$$
$$y(1) = c$$

# *First Order Linear Differential Equations*

## *Example: Exposed water pipe in cyclical ambient temperature*

# *Exposed Water Pipe in Cyclical Ambient Temperature*




$\overline{T}$ = average daily temperature

# *Exposed Water Pipe in Cyclical Ambient Temperature (2)*

**Assuming pipe wall is thin and made of material that is a good heat conductor, by Newton's law of cooling, the heat transferred from air to water is**

$$q = hA(T_\infty - T_W)$$

**where**

$A =$ **Exposed surface area of the pipe**

$h =$ **Convection coefficient**

# *Exposed Water Pipe in Cyclical Ambient Temperature (3)*

**The thermal energy stored in the water is**

$$E = mcT_W$$

**where**

$m =$ **mass of the water**

$c =$ **specific heat of water**

# *Exposed Water Pipe in Cyclical Ambient Temperature (4)*

**Key physical principle:**

$$\frac{dE}{dt} = q$$

**which leads to**

$$mc\frac{d}{dt}\left(\overline{T} + T\right) = hA\left((\overline{T} + T_0 \sin \omega t) - (\overline{T} + T)\right)$$

$$mc\frac{dT}{dt} + (hA)T = (hA)T_0 \sin \omega t$$

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t \qquad \text{where} \qquad \lambda = \frac{hA}{mc}$$

# *Exposed Water Pipe in Cyclical Ambient Temperature (5)*

**How do we solve**

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t \qquad ? \qquad \textbf{Equation 2}$$

**Let's be more inclusive and ask how do we solve the general linear first order nonhomogeneous equation**

$$\frac{dy}{dt} + p(t)y = g(t) \qquad \textbf{Equation 3}$$

$$y(t_0) = y_0$$

# *General Solution to Nonhomogeneous Linear First Order ODEs*

**We begin by searching for an integrating factor $\mu(t)$ that, when multiplied into the equation, turns the left-hand side into**

$$\frac{d}{dt}(\mu(t)y)$$

**Multiplying Equation 3 by $\mu(t)$ :**

$$\mu(t)\frac{dy}{dt}+\mu(t)p(t)y=\mu(t)g(t)$$

**Search for $\mu(t)$ such that left hand side is**

$$\mu(t)\frac{dy}{dt}+\mu(t)p(t)y=\frac{d}{dt}(\mu(t)y)$$

# General Solution to Nonhomogeneous Linear First Order ODEs (2)

**We must have**

$$\mu(t)\frac{dy}{dt}+\mu(t)p(t)y=\frac{d}{dt}(\mu(t)y)=\mu(t)\frac{dy}{dt}+\frac{d\mu}{dt}y$$

**which means that**

$$\frac{d\mu}{dt}=p(t)\mu(t)$$

$$\frac{1}{\mu(t)}\frac{d\mu}{dt}=p(t)$$

# *General Solution to Nonhomogeneous Linear First Order ODEs (3)*

$$\frac{d}{dt}(\ln \mu(t)) = p(t)$$

$$\ln(\mu(t)/\mu(t_0)) = \int_{t_0}^{t} p(u)du$$

$$\mu(t) = \mu(t_0)\exp\int_{t_0}^{t} p(u)du \qquad \textbf{Equation 4}$$

**This is the desired integrating factor.**

**But we can simplify the form.**

# *General Solution to Nonhomogeneous Linear First Order ODEs (4)*

**We do not know what value to assign to $\mu(t_0)$ but it turns out not to matter. (The value cancels out.) So we set**

$$\mu(t_0) = 1$$

**It also suffices to use the indefinite integral form:**

$$\mu(t) = \exp \int^t p(u)du \qquad \textbf{Equation 5}$$

**You should remember, or be able to derive, Equation 5**

# *Exposed Water Pipe in Cyclical Ambient Temperature (6)*

**For the water pipe temperature problem (Equation 2):**

$$\frac{dy}{dt} + p(t)\,y = g(t)$$

**becomes**

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t$$

**so**

$$p(t) = \lambda$$

$$\mu(t) = \exp\left(\int^t p(u)du\right) = \exp\left(\int^t \lambda du\right) = e^{\lambda t}$$

# *Exposed Water Pipe in Cyclical Ambient Temperature (7)*

**Applying the integration factor to Equation 2:**

$$e^{\lambda t}\left(\frac{dT}{dt}+\lambda T\right)=e^{\lambda t}(\lambda T_0 \sin \omega t)$$

$$\frac{d}{dt}\left(e^{\lambda t}T\right)=\lambda T_0 e^{\lambda t}\sin \omega t$$

**Now the value of the integration factor becomes clear:**
**We can solve the problem with an integration:**

$$e^{\lambda t}T(t)-T(0)=\lambda T_0\int_0^t e^{\lambda \tau}\sin \omega \tau d\tau$$

$$T(t)=T(0)e^{-\lambda t}+e^{-\lambda t}\int_0^t e^{\lambda \tau}\sin \omega \tau d\tau$$

# *Exposed Water Pipe in Cyclical Ambient Temperature (9)*

**After performing the integral we have**

$$T(t) = T_I e^{-\lambda t} + \left( \frac{\lambda}{\lambda^2 + \omega^2} \right) T_0 (\lambda \sin(\omega t) - \omega \cos(\omega t) + \omega e^{-\lambda t})$$

**where**

$$T_I = T(0) = T_W(0) - \overline{T}$$

# Inhomogeneous First Order Linear ODEs: In-class Problems

$$\frac{dy}{dt}+\left(\frac{2}{t}\right)y=4$$

$$y(1)=2$$

$$\frac{dy}{dt}+4\left(\frac{e^{4t}-e^{-4t}}{e^{4t}+e^{-4t}}\right)y=e^{3t}$$

$$y(0)=6$$

$$\frac{dy}{dt}-(\tan t)y=\sec t$$

$$y(0)=0$$

# *Homework Assignment 2*

**In text:**

**Read: Chapter 2**

**Work: On course website: Homework Assignment #2 Problems**

**Solutions for Homework Assignment #2 Problems will be provided on course website on (date)**

**Always read over the day's lecture notes and be sure you understand them.**

Differential Equations for Engineers The Essentials 1st edition Kalbaugh Solutions Manual